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# A mixed finite-element method for fourth-order elliptic problems with variable coefficients

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*Abstract:* A new mixed finite-element method for the solution of the Dirichlet problem of fourth-order elliptic partial differential equations with variable coefficients on a convex polygon has been developed in this paper. Biharmonic and bending problems of elastic plates, for which this technique allows a simultaneous approximation to the deflection, components of the change in curvature tensor and the bending and twisting moments, are the particular cases of the problem considered in the paper. Error estimate for the mixed finite-element solution has been given.

*Keywords:* Mixed finite-element method, fourth-order elliptic equations, variable coefficients, anisotropic/orthotropic/isotropic plate problems, error estimates.

## Introduction

It is well known that in the theory of elasticity, the Hu–Washizu variational principle [15,23] (also called 'three-field principle' [12]) and its different modified versions are the most general settings [21,22] for general mixed finite-element methods in the sense that simultaneous and direct approximations to all basic unknown fields are obtained. In plate bending analysis, the Hu–Washizu variational principle or its different variants will allow simultaneous variations of

(i) displacement field $u$,
(ii) change in curvature tensor field ($\kappa_{ij}^*$), and
(iii) bending moment tensor field ($\psi_{ij}$).

Then, a mixed finite-element method based on the Hu–Washizu variational principle or its variants will allow direct and simultaneous approximations to all informations: displacement $u$, change in curvature tensor ($\kappa_{ij}^*$) and bending moment tensor ($\psi_{ij}$) required in the bending analysis of anisotropic/orthotropic/isotropic plates with variable/constant thickness such that no further computations will be necessary. Such a mixed finite-element method has been developed in [2] and [4] for the Dirichlet problem of fourth-order elliptic partial differential equations with variable/constant coefficients on a convex polygon, the bending problems of clamped thin elastic anisotropic/orthotropic/isotropic plates with variable/constant thickness being particular cases, for which the continuity of the normal bending moment $M_n(\Phi)$ across intertriangular boundaries is *not* assumed in the construction of the space of variationally admissible bending moment tensor field $\Phi = (\phi_{ij})$. This suggests to develop a new mixed finite-element method for this Dirichlet problem of the fourth-order elliptic equations with variable coefficients based on another modified version of the Hu–Washizu variational principle which is obtained by incorporating this physically meaningful and important constraint of 'continuity' of the normal bending moment $M_n(\Phi)$ across intertriangular boundaries in the definition of the space $\mathbf{V}$ of variationally admissible bending moment tensor field $\Phi = (\phi_{ij})$. This paper contains new interesting results in this direction. For other mixed methods for this Dirichlet problem based on different two-field variational principles [12,20], we refer to [5]–[9].

## 1. Notations

Let $\Omega$ be a convex polygon with boundary $\Gamma$ in $\mathbb{R}^2$ and $H^m(\Omega)$ be the usual Sobolev space [1,19] of integral order $m \geqslant 0$ equipped with inner product $\langle \cdot, \cdot \rangle_{m,\Omega}$ norm $\|\cdot\|_{m,\Omega}$ and semi-norm $|\cdot|_{m,\Omega}$ such that $H^0(\Omega) \equiv L^2(\Omega)$,

$$H_0^1(\Omega) = \{v : v \in H^1(\Omega),\ \gamma_0 v = v|_\Gamma = 0\},$$

$$H_0^2(\Omega) = \{v : v \in H^2(\Omega),\ \gamma_0 v = v|_\Gamma = 0,\ \gamma_1 v = (\partial v/\partial n)|_\Gamma = 0\}, \tag{1.1}$$

where $\partial v/\partial n$ is the derivative of $v$ in the direction of the exterior normal to the boundary $\Gamma$; $\gamma_k : H^m(\Omega) \to H^{m-k-1/2}(\Gamma)$ are trace operators [1,19], $m = 1, 2$; $k = 0,\ m-1$; $H^{1/2}(\Gamma)$, $H^{3/2}(\Gamma)$ being the fractional order Sobolev spaces of functions on $\Gamma$; $\overline{D(\Omega)} \equiv H_0^m(\Omega)$ in the norm topology of $H^m(\Omega)$, $m \geqslant 1$, $D(\Omega)$ being the space of test functions.

Let $W_0^{1,p}(\Omega)$, $p > 2$, be the Sobolev space [1,19], defined by

$$\forall p > 2, \quad W_0^{1,p}(\Omega) = \{v : v \in W^{1,p}(\Omega),\ v|_\Gamma = 0\}, \tag{1.2}$$

such that

$$H_0^2(\Omega) \hookrightarrow W_0^{1,p}(\Omega) \hookrightarrow H_0^1(\Omega) \quad (p > 2) \tag{1.3}$$

with dense, continuous injections; and

$$H_0^2(\Omega) \hookrightarrow W_0^{1,p}(\Omega) \hookrightarrow C^0(\overline{\Omega}) \quad (p > 2), \tag{1.4}$$

(1.3) and (1.4) being the consequences of the Sobolev's imbedding theorem [1,19].

## 2. The continuous variational problem

To the Dirichlet problem (P) defined by: For given $f \in L^2(\Omega)$, find $u$ such that

$$\text{(P)} \qquad \Lambda u = f \ \text{ in } \Omega, \quad u|_\Gamma = (\partial u/\partial n)|_\Gamma = 0, \tag{2.1}$$

where

$$(\Lambda u)(x) \equiv \frac{\partial^2}{\partial x_k\, \partial x_l}\left(a_{ijkl}(x) \frac{\partial^2 u}{\partial x_i\, \partial x_j}\right) \equiv \left(a_{ijkl}(x) u_{,ij}\right)_{,kl} \ \text{ in } \Omega \tag{2.2}$$

(*in* (2.2) *and also in the sequel, the Einstein summation convention has been followed*), we associate the following Galerkin variational problem ($P_G$) defined by: Find $u \in H_0^2(\Omega)$ such that

$$(P_G) \quad a(u, v) = l(v) \quad \forall v \in H_0^2(\Omega), \tag{2.3}$$

where the continuous, symmetric, bilinear form a $(\cdot, \cdot)$ and the continuous linear form $l(\cdot)$ are defined by: $\forall v, w \in H_0^2(\Omega)$,

$$a(v, w) = \langle \Lambda v, w \rangle_{0,\Omega} = \int_\Omega a_{ijkl} v_{,ij} w_{,kl} \, d\Omega, \tag{2.4}$$

$$l(v) = \langle f, v \rangle_{0,\Omega} = \int_\Omega fv \, d\Omega \quad \forall v \in H_0^2(\Omega); \tag{2.5}$$

the coefficients $a_{ijkl}$ satisfy the following conditions: $\forall i, j, k, l = 1, 2$,

(A1) $a_{ijkl} \in C^0(\bar{\Omega})$, $a_{ijkl}(x) = a_{klij}(x)$ $\forall x \in \bar{\Omega}$; but without loss of generality [3], we can always assume that $\forall i, j, k, l = 1, 2$;

(A1′) $a_{ijkl}(x) = a_{klij}(x) = a_{ijlk}(x) = a_{jilk}(x)$ $\forall x \in \bar{\Omega}$ and *in the sequeli* we *will assume that* (A1′) *holds always*;

(A2) $\forall v \in H_0^2(\Omega)$, $a(v, v) \geqslant \alpha \| v \|_{2,\Omega}^2$ for some $\alpha > 0$; i.e. $a(\cdot, \cdot)$ defined by (2.4) is $H_0^2(\Omega)$—elliptic;

(A2′) $\forall \xi = (\xi_{11}, \xi_{12}, \xi_{21}, \xi_{22}) \in \mathbb{R}^4$ with $\xi_{12} = \xi_{21}$, $\exists \alpha_0 > 0$ such that $\forall x \in \bar{\Omega}$, $a_{ijkl}(x) \xi_{ij} \xi_{kl} \geqslant \alpha_0 \| \xi \|_{\mathbb{R}^4}^2$.

**Remark 2.1.** If (A1) holds, but (A1′) does *not* hold, i.e. $a_{ijkl} \neq a_{jikl}$ or $a_{ijkl} \neq a_{ijlk}$ for some $i, j, k, l = 1, 2$, then define new coefficients

$$\bar{a}_{ijkl} = \tfrac{1}{4}(a_{ijkl} + a_{ijlk} + a_{jikl} + a_{jilk}) \quad \text{(see (2.8))}, \tag{2.6}$$

for which (A1′) will hold,

$$a(u, v) = \int_\Omega a_{ijkl} u_{,ij} v_{,kl} \, d\Omega = \int_\Omega \bar{a}_{ijkl} u_{,ij} v_{,kl} \, d\Omega. \tag{2.7}$$

**Remark 2.2.** In [3], sufficient conditions for (A2′) to hold have been given. Then, it has also been shown that (A2′) ⇒ (A2).

**Theorem 2.1** [3]. Under the assumptions (A1)–(A2), the problem ($P_G$) has a unique solution.

*Examples*

Here, we will consider only the biharmonic problem (i.e. Dirichlet problem of the biharmonic operator $\Delta\Delta$) and bending problems of clamped elastic plates with variable/constant thickness.

*I. The biharmonic problem*

For $a_{ijkl} = \frac{1}{4}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} + \delta_{li}\delta_{kj} + \delta_{lj}\delta_{ki})$, i.e.

$$a_{iiii} = 1, \quad a_{1212} = a_{1221} = a_{2112} = a_{2121} = \tfrac{1}{2}, \qquad a_{ijkl} = 0 \quad \text{otherwise}, \tag{2.8}$$

$$\Lambda u \equiv \Delta\Delta u \text{ and } a(u, v) = \int_\Omega u_{,ij} v_{,ij} \, d\Omega, \tag{2.9}$$

for which (A1), (A1′), (A2), (A2′) hold [3], and the existence and uniqueness of solution of the corresponding ($P_G$) follow from Theorem (2.1).

**Remark 2.3.** Even with $a_{ijkl} = \delta_{ik}\delta_{jl}$ one gets $\Lambda u \equiv \Delta\Delta u$, but for this choice, (A1′) does *not* hold, although (A1) holds, since $a_{ijlk} \neq a_{ijkl}$, $a_{ijkl} \neq a_{jikl}$. Hence, following (2.6), (2.8) has been chosen so that (A1′) holds [3].

**Remark 2.4.** $a_{ijkl} = \delta_{ij}\delta_{kl}$ also gives $\Lambda u \equiv \Delta\Delta u$, but $a(u, v) = \int_\Omega \Delta u \, \Delta v \mathrm{d}\Omega$, which differs from that in (2.9). For this choice of $a_{ijkl}$, (A1), (A1′), (A2) hold [3] and the corresponding ($P_G$) has a unique solution, but (A2′) does *not* hold. Hence, this choice will *not* be considered by us in the sequel.

*II. Bending problems of clamped plates with variable thickness*

(i) Anisotropic case [3]:

$$a_{iiii} = D_{ii}, \quad a_{1212} = a_{1221} = a_{2121} = a_{2112} = D_{66}, \quad a_{1112} = a_{1211} = a_{2111} = a_{1121} = D_{16},$$
$$a_{1222} = a_{2122} = a_{2212} = a_{2221} = D_{26}, \ a_{2211} = a_{1122} = D_{12}, \tag{2.10}$$

where $D_{ii} = D_{ii}(x)$, $D_{12} = D_{12}(x)$, $D_{i6} = D_{i6}(x)$, $D_{66} = D_{66}(x)$ are 'rigidities' [18], for which (A1), (A1′), (A2), (A2′) hold [3];

$$\begin{aligned}\Lambda u \equiv\ & (D_{11}u_{,11} + 2D_{16}u_{,12} + D_{12}u_{,22})_{,11} + 2(D_{16}u_{,11} + 2D_{66}u_{,12} + D_{26}u_{,22})_{,12} \\ & + (D_{12}u_{,11} + 2D_{26}u_{,12} + D_{22}u_{,22})_{,22};\end{aligned} \tag{2.11}$$

$$\begin{aligned}a(u, v) = \int_\Omega \big[ & (D_{11}u_{,11} + 2D_{16}u_{,12} + D_{12}u_{,22})v_{,11} \\ & + 2(D_{16}u_{,11} + 2D_{66}u_{,12} + D_{26}u_{,22})v_{,12} \\ & + (D_{12}u_{,11} + 2D_{26}u_{,12} + D_{22}u_{,22})v_{,22}\big] \, \mathrm{d}\Omega;\end{aligned} \tag{2.12}$$

(ii) Orthotropic case [3]:

$$a_{iiii} = D_i, \quad a_{1122} = a_{2211} = D_{12} = \nu_1 D_2 = \nu_2 D_1, \quad a_{1212} = a_{1221} = a_{2112} = a_{2121} = D_t,$$
$$a_{ijkl} = 0 \text{ otherwise}, \tag{2.13}$$

where $D_i = E_i t^3/(12(1 - \nu_1\nu_2))$, $D_t = Gt^3/12$, $E_i$ = Young's moduli, $\nu_i$ = Poisson's coefficients, $G$ = shearing modulus, $t = t(x_1, x_2)$ = thickness of plate at $(x_1, x_2) \in \overline{\Omega}$ with

$$t_0 = \min_{x \in \overline{\Omega}} t(x_1, x_2) > 0, \quad t \in C^0(\overline{\Omega}), \tag{2.14}$$

for which (A1), (A1′), (A2), (A2′) hold [3];

$$\Lambda u \equiv (D_1 u_{,11} + \nu_2 D_1 u_{,22})_{,11} + 4(D_t u_{,12})_{,12} + (\nu_1 D_2 u_{,11} + D_2 u_{,22})_{,22}; \tag{2.15}$$

$$a(u, v) = \int_\Omega \big[(D_1 u_{,11} + \nu_2 D_1 u_{,22})v_{,11} + 4D_t u_{,12} v_{,12} + (\nu_1 D_2 u_{,11} + D_2 u_{,22})v_{,22}\big] \, \mathrm{d}\Omega; \tag{2.16}$$

(iii) Isotropic case [3]:

$$a_{iiii} = D, \quad a_{1122} = a_{2211} = \nu D, \quad a_{1212} = a_{1221} = a_{2112} = a_{2121} = D(1-\nu)/2,$$
$$a_{ijkl} = 0 \text{ otherwise}, \tag{2.17}$$

$D = Et^3/12(1-\nu^2)$ is the bending rigidity with Young's modulus $E$, Poisson's coefficient $\nu$, for which (A1), (A1′), (A2), (A2′) hold [3];

$$\Lambda u \equiv (D(u_{,11} + \nu u_{,22}))_{,11} + 2(D(1-\nu)u_{,12})_{,12} + (D(\nu u_{,11} + u_{,22}))_{,22}; \tag{2.18}$$

$$a(u, v) = \int_\Omega D[(u_{,11} + \nu u_{,22})v_{,11} + 2(1-\nu)u_{,12}v_{,12} + (\nu u_{,11} + u_{,22})v_{,22}] \, d\Omega. \tag{2.19}$$

In all the three cases (i)–(iii), the existence and uniqueness of solution of the corresponding problem ($P_G$) follow from Theorem 2.1.

## 3. Mixed variational formulation

A new mixed variational formulation for the problem ($P_G$) will be constructed now, the admissible spaces for which are infinite-dimensional ones. Then, to this mixed variational formulation we will associate a mixed finite-element problem defined on finite-dimensional spaces. Hence, we are to introduce first of all admissible infinite-dimensional spaces of the mixed variational formulation. For this we introduce an admissible triangulation $T_h$ of $\bar{\Omega} = \Omega \cup \Gamma$ into closed triangles $T$ with boundary $\partial T$ (which will be finally also the triangulation of the mixed finite element scheme). Let $\Phi = (\phi_{ij})_{1 \leqslant i,j \leqslant 2}$ be a symmetric tensor valued function with $\phi_{ij} = \phi_{ji} \in H^1(T)$ $\forall T \in T_h$, $\forall i$, $j = 1, 2$. Then, define $M_n(\Phi) \in L^2(\partial T)$ and $M_{nt}(\Phi) \in L^2(\partial T)$ along $\partial T$ by:

$$M_n(\Phi) = \phi_{ij} n_i n_j, \qquad M_{nt}(\Phi) = \phi_{ij} n_i t_j,$$
$$\left(Q_n(\Phi) = \phi_{ij,i} n_j \quad \text{for } \phi_{ij} \in H^2(T) \quad \forall i, j = 1, 2\right), \tag{3.1}$$

where $\boldsymbol{n} = (n_1, n_2)$ and $\boldsymbol{t} = (t_1, t_2) = (n_2, -n_1)$ are the unit normal vector exterior to $\partial T$ and unit tangent vector along $\partial T$ respectively.

**Remark 3.1.** When $\Lambda$ is the elastic plate bending operator in (2.11), (2.15) or (2.18), $M_n(\Phi)$ and $M_{nt}(\Phi)$ denote 'normal bending moment' and 'twisting bending moment' along $\partial T$ respectively due to admissible bending moment tensor field $\Phi = (\phi_{ij})_{1 \leqslant i,j \leqslant 2}$. Then, $M_n(\Phi)$ is called '*continuous*' at the interelement boundaries of $T_h$, iff $\forall$ pair of adjacent triangles $T_1, T_2 \in T_h$ with a common side $\partial T_1 \cap \partial T_2$, $\partial T_i$ being the boundary of $T_i$,

$$M_{n_1}(\Phi|_{T_1}) = M_{n_2}(\Phi|_{T_2}) \quad \text{on } T_1 \cap T_2 = \partial T_1 \cap \partial T_2, \tag{3.2}$$

$\boldsymbol{n}_i$ being the unit normal exterior to $T_i$ ($i = 1, 2$) and $\Phi = (\phi_{ij})$ being an admissible symmetric bending moment tensor field on $\Omega$.
*This definition of 'continuity' of $M_n(\Phi)$ at the interelement boundaries of $T_h$ will be understood throughout the paper even when $\Lambda$ is not a plate bending operator.*

$Q_n(\Phi)$ in (3.1) denotes transverse shear along $\partial T$.

Define

$$\mathbf{M} = \left\{\boldsymbol{\kappa}\colon\ \boldsymbol{\kappa} = (\kappa_{ij}),\ 1 \leqslant i,\ j \leqslant 2,\ \kappa_{12} = \kappa_{21},\ \kappa_{ij} \in L^2(\Omega)\right\},$$

$$\|\boldsymbol{\kappa}\|^2_{\mathbf{M}} = \|\boldsymbol{\kappa}\|^2_{0,\Omega} = \sum_{i,j=1}^{2} \|\kappa_{ij}\|^2_{0,\Omega} \quad \forall \boldsymbol{\kappa} = (\kappa_{ij}) \in \mathbf{M}; \tag{3.3}$$

$$V = H^1_0(\Omega) \times \mathbf{M}, \quad \forall \boldsymbol{v} = (v,\ \boldsymbol{\kappa}) \in V, \quad \|\boldsymbol{v}\|^2_V = \|(v,\ \boldsymbol{\kappa})\|^2_V = \|v\|^2_{1,\Omega} + \|\boldsymbol{\kappa}\|^2_{\mathbf{M}}; \tag{3.4}$$

$$V_p = W^{1,p}_0(\Omega) \times \mathbf{M} \quad (p > 2), \quad \forall \boldsymbol{v} = (v,\ \boldsymbol{\kappa}) \in V_p,$$

$$\|\boldsymbol{v}\|^2_{V_p} = \|(v,\ \boldsymbol{\kappa})\|^2_{V_p} = \|v\|^2_{W^{1,p}(\Omega)} + \|\boldsymbol{\kappa}\|^2_{\mathbf{M}}; \tag{3.5}$$

$$\mathbf{W} = \left\{\boldsymbol{\Phi}\colon\ \boldsymbol{\Phi} = (\phi_{ij}) \in \mathbf{M},\ \phi_{ij}|_T \in H^1(T)\ \forall T \in T_h,\right.$$
$$\left. M_n(\boldsymbol{\Phi}) \text{ satisfies (3.2) at interelement boundaries}\right\}, \quad \|\boldsymbol{\Phi}\|^2_{\mathbf{W}} = \sum_{T \in T_h} \|\phi_{ij}\|^2_{1,T}. \tag{3.6}$$

The imbeddings $\mathbf{W} \hookrightarrow \mathbf{M}$, $V_p \hookrightarrow V$ are continuous with:

$$\|\boldsymbol{\Phi}\|_{\mathbf{M}} \leqslant \|\boldsymbol{\Phi}\|_{\mathbf{W}} \quad \forall \boldsymbol{\Phi} \in \mathbf{W}, \quad \|\boldsymbol{v}\|_V \leqslant \sigma_1 \|\boldsymbol{v}\|_{V_p} \quad \forall \boldsymbol{v} \in V_p, \quad \sigma_1 > 0, \tag{3.7}$$

$V_p \hookrightarrow V$ being dense too ($p > 2$).

For the problem ($\mathrm{P_G}$), we can now construct the problem ($\mathrm{Q}^*$) of the mixed variational formulation under consideration as follows: Find $(u,\ \boldsymbol{\kappa}^*,\ \boldsymbol{\Psi}) \in W^{1,p}_0(\Omega) \times \mathbf{M} \times \mathbf{W}$ ($p > 2$) such that

$$a(\boldsymbol{\kappa}^*,\ \boldsymbol{\eta}) - [\boldsymbol{\Psi},\ \boldsymbol{\eta}]_{0,\Omega} = 0 \quad \forall \boldsymbol{\eta} = (\eta_{ij}) \in \mathbf{M}, \tag{3.8}$$

$$(\mathrm{Q}^*) \quad b^*(u,\ \boldsymbol{\Phi}) - [\boldsymbol{\kappa}^*,\ \boldsymbol{\Phi}]_{0,\Omega} = 0 \quad \forall \boldsymbol{\Phi} = (\phi_{ij}) \in \mathbf{W}, \tag{3.9}$$

$$b^*(v,\ \boldsymbol{\Psi}) = \langle f,\ v\rangle_{0,\Omega} \quad \forall v \in W^{1,p}_0(\Omega), \quad p > 2, \tag{3.10}$$

where the continuous symmetric bilinear form $\tilde{a}(\cdot,\ \cdot)\colon \mathbf{M} \times \mathbf{M} \to \mathbb{R}$ is defined by

$$\forall \boldsymbol{\kappa} = (\kappa_{ij}) \in \mathbf{M}, \qquad \boldsymbol{\eta} = (\eta_{ij}) \in \mathbf{M},$$

$$\tilde{a}(\boldsymbol{\kappa},\ \boldsymbol{\eta}) = \int_\Omega a_{ijkl}\kappa_{ij}\eta_{kl}\ \mathrm{d}\Omega = \tilde{a}(\boldsymbol{\eta},\ \boldsymbol{\kappa}); \tag{3.11}$$

the continuous bilinear form $b^*(\cdot,\ \cdot)\colon W^{1,p}_0(\Omega) \times \mathbf{W} \to \mathbb{R}$ ($p > 2$) is defined by:

$$\forall v \in W^{1,p}_0(\Omega), \qquad \forall \boldsymbol{\Phi} = (\phi_{ij}) \in \mathbf{W},$$

$$b^*(v,\ \boldsymbol{\Phi}) = \sum_{T \in T_h} \left[-\int_T v_{,i}\phi_{ij,j}\ \mathrm{d}\Omega + \int_{\partial T} M_{nt}(\boldsymbol{\Phi}) \frac{\partial v}{\partial t}\,\mathrm{d}s\right], \tag{3.12}$$

$$[\boldsymbol{\Phi},\ \boldsymbol{\kappa}]_{0,\Omega} = [\boldsymbol{\kappa},\ \boldsymbol{\Phi}]_{0,\Omega} = \int_\Omega \phi_{ij}\kappa_{ij}\ \mathrm{d}\Omega \quad \forall \boldsymbol{\Phi} \in \mathbf{W}, \quad \forall \boldsymbol{\kappa} \in \mathbf{M}. \tag{3.13}$$

This new mixed variational formulation ($\mathrm{Q}^*$) in (3.8)–(3.13), which corresponds to a modified version of the three-field principle also called Hu–Washizu variational principle [15,20,21,22,23], can be rewritten in the standard form (Q) suitable for its mathematical analysis as follows: Find $(\boldsymbol{u}, \Psi) \in V_p \times \mathbf{W}(p > 2)$ such that

$$\text{(Q)} \quad \begin{aligned} &A(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, \Psi) = F(\boldsymbol{v}) \quad \forall \boldsymbol{v} = (v, \eta) \in V_p && (3.14)\\ &b(\boldsymbol{u}, \Phi) = 0 \quad \forall \Phi \in \boldsymbol{W}, && (3.15) \end{aligned}$$

where $\boldsymbol{u} = (u, \kappa^*) \in V_p$ with $u \in W_0^{1,p}(\Omega)$, $\kappa^* = (\kappa_{ij}^*) \in \mathbf{M}$; the new continuous, symmetric bilinear form $A(\cdot, \cdot)$ on $V_p \times V_p$, $(p > 2)$ and the continuous bilinear form $b(\cdot, \cdot)$ on $V_p \times \mathbf{W}$ are defined by: $\forall \boldsymbol{v} = (v, \eta) \in V_p$, $\forall \boldsymbol{w} = (w, \zeta) \in V_p$, $(p > 2)$

$$A(\boldsymbol{v}, \boldsymbol{w}) = A(\boldsymbol{w}, \boldsymbol{v}) = \tilde{a}(\eta, \zeta) = \int_\Omega a_{ijkl}\eta_{ij}\zeta_{kl}\, \mathrm{d}\Omega; \tag{3.16}$$

$$b(\boldsymbol{v}, \Phi) = b^*(v, \Phi) - [\eta, \Phi]_{0,\Omega} \quad \forall \boldsymbol{v} = (v, \eta) \in V_p, \quad \forall \Phi \in \mathbf{W} \tag{3.17}$$

$$= \sum_{T \in T_h} \left[ -\int_T v_{,i}\phi_{ij,j}\, \mathrm{d}T + \int_{\partial T} M_{nt}(\Phi) \frac{\partial v}{\partial t} \mathrm{d}s \right] - \int_\Omega \eta_{ij}\phi_{ij}\, \mathrm{d}\Omega; \tag{3.18}$$

the continuous linear form $F(\cdot)$ on $V_p$ is defined by:

$$F(\boldsymbol{v}) = \langle f, v \rangle_{0,\Omega} \quad \forall \boldsymbol{v} = (v, \eta) \in V_p. \tag{3.19}$$

In fact, equation (3.9) $\Rightarrow$ (3.15) with $\boldsymbol{u} = (u, \kappa^*) \in V_p$. Then, from (3.8) and (3.10) we have

$$\tilde{a}(\kappa^*, \eta) - [\eta, \Psi]_{0,\Omega} + b^*(v, \Psi) = \langle f, v \rangle_{0,\Omega} \quad \forall v \in W_0^{1,p}(\Omega), \quad \forall \eta \in \mathbf{M}$$

$$\Rightarrow A(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, \Psi) = F(\boldsymbol{v}) \quad \forall \boldsymbol{v} = (v, \eta) \in V_p,$$

since $\forall \boldsymbol{v} = (v, \eta) \in V_p$, $b(\boldsymbol{v}, \Psi) = b^*(v, \Psi) - [\eta, \Psi]_{0,\Omega}$ and $F(\boldsymbol{v}) = \langle f, v \rangle_{0,\Omega}$, and $\tilde{a}(\kappa^*, \eta) = A(\boldsymbol{u}, \boldsymbol{v})$ with $\boldsymbol{u} = (u, \kappa^*) \in V_p$.

Thus, we have shown that $(\mathrm{Q}^*) \Rightarrow (\mathrm{Q})$. Again, proceeding in the reverse way, we can show $(\mathrm{Q}) \Rightarrow (\mathrm{Q}^*)$.

*Hence, the problems* (Q) *and* (Q*) *are equivalent*. Thus, we are free to choose any one of these two equivalent problems (Q) and (Q*). For the sake of convenience in *mathematical analysis we will consider the problem* (Q) *in the sequel unless stated otherwise*. Hence, we are going to prove now certain important properties of $A(\cdot, \cdot)$ and $b(\cdot, \cdot)$. For this we introduce the subspace $\boldsymbol{Z}$ of $V_p$, $(p > 2)$ defined by

$$\boldsymbol{Z} = \{\boldsymbol{v}\colon \boldsymbol{v} = (v, \kappa) \in V_p,\ b(\boldsymbol{v}, \Phi) = 0\ \forall \Phi \in \mathbf{W}\}. \tag{3.20}$$

Then, $\boldsymbol{Z}$ itself is a Hilbert space equipped with the norm topology induced by $V$, i.e.

$$\|\boldsymbol{v}\|_{Z} = \|\boldsymbol{v}\|_V = \left( \|v\|_{1,\Omega}^2 + \|\kappa\|_{0,\Omega}^2 \right)^{1/2} \quad \forall \boldsymbol{v} = (v, \kappa) \in \boldsymbol{Z}. \tag{3.21}$$

**Theorem 3.1.** *Let* $\boldsymbol{v} = (v, \kappa) \in V_p$, $p > 2$. *Then*, $\boldsymbol{v} \in \boldsymbol{Z} \Leftrightarrow$ (i) $v \in H_0^2(\Omega)$, (ii) $\kappa = (\kappa_{ij})$ *with* $\kappa_{ij} = v_{,ij}$, $1 \leqslant i, j \leqslant 2$.

**Proof.** Let $\boldsymbol{v} \in \boldsymbol{Z}$. Then, $b(\boldsymbol{v}, \boldsymbol{\Phi}) = 0$ $\forall \boldsymbol{\Phi} \in \mathbf{W}$. Define $\boldsymbol{\Phi}^* = (\phi\delta_{ij})$, $1 \leqslant i,\ j \leqslant 2$ with $\phi \in D(\Omega)$. Then, $M_n(\boldsymbol{\Phi}^*) = \phi$, $M_{nt}(\boldsymbol{\Phi}) = 0$ $\forall T \in T_h$. Thus,

$$\boldsymbol{\Phi}^* \in \mathbf{W} \quad \text{and} \quad b(\boldsymbol{v}, \boldsymbol{\Phi}^*) = 0$$

$$\Rightarrow \int_\Omega v_{,i}\phi_{,i}\ \mathrm{d}\Omega = -\int_\Omega \kappa_{ii}\phi\ \mathrm{d}\Omega \quad \forall \phi \in D(\Omega)$$

$$\Rightarrow \int_\Omega \operatorname{grad} v \cdot \operatorname{grad} \phi\ \mathrm{d}\Omega = -\int_\Omega \kappa_{ii}\phi\ \mathrm{d}\Omega \quad \forall \phi \in H_0^1(\Omega) = \overline{D(\Omega)}$$

$$\Rightarrow v \in H^2(\Omega) \cap H_0^1(\Omega),$$

since $\Gamma$ is a convex polygon [17], and

$$\Delta v = \kappa_{11} + \kappa_{22} \in L^2(\Omega). \tag{3.22}$$

Now, choosing $\boldsymbol{\Phi}^* = (\phi\delta_{ij})$, $1 \leqslant i, j \leqslant 2$ with $\phi \in D(\overline{\Omega})$, we can show

$$\boldsymbol{\Phi}^* \in \mathbf{W}, \qquad M_{nt}(\boldsymbol{\Phi}^*) = 0,$$

and

$$\int_\Omega \operatorname{grad} v \cdot \operatorname{grad} \phi\ \mathrm{d}\Omega = -\int_\Omega \kappa_{ii}\phi\ \mathrm{d}\Omega \quad \forall \phi \in D(\overline{\Omega})$$

$$\Rightarrow \text{for } v \in H^2(\Omega) \cap H_0^1(\Omega),$$

$$-\int_\Omega (\Delta v)\phi\ \mathrm{d}\Omega + \int_\Gamma \frac{\partial v}{\partial n}\phi\ \mathrm{d}\Gamma = -\int_\Omega (\Delta v)\phi\ \mathrm{d}\Omega$$

$$\forall \phi \in H^1(\Omega) \equiv \overline{D(\overline{\Omega})} \quad (\text{by } (3.22))$$

$$\Rightarrow \int_\Gamma \frac{\partial v}{\partial n}\phi\ \mathrm{d}\Gamma = 0 \quad \forall \phi \in H^1(\Omega)$$

$$\Rightarrow \forall(\gamma_0\phi) \in H^{1/2}(\Gamma), \quad \langle \gamma_1 v, \gamma_0\phi \rangle_{0,\Gamma} = 0 \quad \text{for } \gamma_1 v \in H^{1/2}(\Gamma)$$

$$\Leftrightarrow \gamma_1 v = (\partial v/\partial n)\,|_\Gamma = 0 \quad \text{with } v \in H^2(\Omega) \cap H_0^1(\Omega) \Rightarrow v \in H_0^2(\Omega).$$

But

$$\boldsymbol{v} \in \boldsymbol{Z} \Rightarrow v \in H_0^2(\Omega) \Rightarrow \sum_{T \in T_h} \int_{\partial T} M_n(\boldsymbol{\Phi}) \frac{\partial v}{\partial n}\,\mathrm{d}s = 0 \quad \forall \boldsymbol{\Phi} \in \mathbf{W}, \tag{3.23}$$

since $M_n(\boldsymbol{\Phi})$ is 'continuous' at the interelement boundaries and $(\partial v/\partial n)\,|_{\partial T} \in H^{1/2}(\partial T)$. Then,

$$\boldsymbol{v} \in \boldsymbol{Z} \Rightarrow b(\boldsymbol{v}, \boldsymbol{\Phi}) = 0\ \forall \boldsymbol{\Phi} \in \mathbf{W}$$

$$\Rightarrow \int_\Omega (-\kappa_{ij} + v_{,ij})\phi_{ij}\ \mathrm{d}\Omega - \sum_{T \in T_h} \int_{\partial T} M_n(\boldsymbol{\Phi}) \frac{\partial v}{\partial n}\,\mathrm{d}s = 0$$

$$\Rightarrow \forall \boldsymbol{\Phi} \in \mathbf{W}, \quad \int_\Omega (-\kappa_{ij} + v_{,ij})\phi_{ij}\ \mathrm{d}\Omega = 0 \quad \Leftrightarrow \quad \kappa_{ij} = v_{,ij} \quad \forall i, j = 1, 2.$$

Conversely, let $\boldsymbol{v}=(v, \boldsymbol{\kappa}) \in \boldsymbol{V}_p$ with $v \in H_0^2(\Omega)$ and $\kappa_{ij}=v_{,ij}$ $\forall i, j=1, 2$. Then, from (3.23)

$$\forall \boldsymbol{\Phi} \in \mathbf{W}, \quad \int_\Omega (v_{,ij}-\kappa_{ij})\phi_{ij}\,\mathrm{d}\Omega - \sum_{T \in T_h} \int_{\partial T} M_n(\boldsymbol{\Phi}) \frac{\partial v}{\partial n}\,\mathrm{d}s = 0$$

$$\Rightarrow \forall \boldsymbol{\Phi} \in \mathbf{W}, \sum_{T \in T_h} \left( -\int_T (\kappa_{ij}\phi_{ij} + v_{,i}\phi_{ij,j})\,\mathrm{d}\Omega + \int_{\partial T} M_{nt}(\boldsymbol{\Phi}) \frac{\partial v}{\partial t}\,\mathrm{d}s \right) = 0$$

$$\Rightarrow b(\boldsymbol{v}, \boldsymbol{\Phi}) = 0 \ \forall \boldsymbol{\Phi} \in \mathbf{W} = \boldsymbol{v} \in \boldsymbol{Z}. \qquad \square$$

**Theorem 3.2.** *The mapping*

$$\boldsymbol{v}=(v, \boldsymbol{\kappa}) \in \boldsymbol{Z} \to \|\boldsymbol{\kappa}\|_{0,\Omega} \tag{3.24}$$

*defines a norm in* $\boldsymbol{Z}$ *equivalent to the original norm* (3.21) *induced by* $\boldsymbol{V}$, *i.e.*

$$\exists \alpha_1, \alpha_2 > 0 \quad \text{such that } \forall \boldsymbol{v}=(v, \boldsymbol{\kappa}) \in \boldsymbol{Z}, \ \alpha_1 \|\boldsymbol{v}\|_{\boldsymbol{V}} \leqslant \|\boldsymbol{\kappa}\|_{0,\Omega} \leqslant \alpha_2 \|\boldsymbol{v}\|_{\boldsymbol{V}}. \tag{3.25}$$

**Proof.** It is well known that $\exists \alpha > 0$ such that $\forall v \in H_0^2(\Omega)$, $\|v\|_{2,\Omega} \leqslant \alpha |v|_{2,\Omega}$, from which the result follows on the application of the Theorem (3.1) with $\alpha_1 = 1/\alpha$, $\alpha_2 = 1$. $\square$

Since $\boldsymbol{V}_p$ $(p>2)$ is a dense subspace of $\boldsymbol{V}$, $A(\cdot, \cdot)$ defined on $\boldsymbol{V}_p \times \boldsymbol{V}_p$ is given a continuous, linear extension to $\boldsymbol{V} \times \boldsymbol{V}$, which is still denoted by $A(\cdot, \cdot)$.

**Theorem 3.3.** (i) $A(\cdot, \cdot)$ *is* $\boldsymbol{Z}$*-elliptic, i.e.*

$$A(\boldsymbol{v}, \boldsymbol{v}) \geqslant \alpha \|\boldsymbol{v}\|_{\boldsymbol{V}}^2 \ \forall \boldsymbol{v} \in \boldsymbol{Z}, \quad \alpha > 0; \tag{3.26}$$

(ii)
$$\sup_{\boldsymbol{v} \in \boldsymbol{V}_p} \frac{b(\boldsymbol{v}, \boldsymbol{\Phi})}{\|\boldsymbol{v}\|_{\boldsymbol{V}_p}} \geqslant \|\boldsymbol{\Phi}\|_{0,\Omega} \quad \forall \boldsymbol{\Phi} \in \mathbf{W}; \tag{3.27}$$

(iii) $\exists \sigma_1 > 0$ *such that*

$$\sup_{\boldsymbol{\Phi} \in \mathbf{W}} \frac{b^*(v, \boldsymbol{\Phi})}{\|\boldsymbol{\Phi}\|_{\mathbf{W}}} \geqslant \sigma_1 \|v\|_{1,\Omega} \quad \forall v \in W_0^{1,p}(\Omega) \quad (p>2), \tag{3.28}$$

*where* $b^*(\cdot, \cdot)$ *is defined in* (3.12).

**Proof.** (i) $\boldsymbol{v}=(v, \boldsymbol{\kappa}) \in \boldsymbol{Z}$ with $\boldsymbol{\kappa}=(\kappa_{ij}) \Rightarrow v \in H_0^2(\Omega)$, $\kappa_{ij}=v_{,ij}$ $\forall i, j=1, 2$ (by Theorem 3.1). Hence,

$$\forall \boldsymbol{v}=(v, \boldsymbol{\kappa}) \in \boldsymbol{Z}, \quad A(\boldsymbol{v}, \boldsymbol{v}) = \int_\Omega a_{ijkl} v_{,ij} v_{,kl}\,\mathrm{d}\Omega = a(v, v) \geqslant \alpha \|v\|_{2,\Omega}^2 = \alpha \|\boldsymbol{v}\|_{\boldsymbol{V}}^2,$$

since $a(\cdot, \cdot)$ is $H_0^2(\Omega)$-elliptic by (A2).

(ii)
$$\forall \boldsymbol{\Phi} \in \mathbf{W}, \quad \sup_{\boldsymbol{v} \in \boldsymbol{V}_p} \frac{b(\boldsymbol{v}, \boldsymbol{\Phi})}{\|\boldsymbol{v}\|_{\boldsymbol{V}_p}} \geqslant \operatorname*{Sup}_{(0,\boldsymbol{\kappa}) \in \boldsymbol{V}_p} \frac{b((0, \boldsymbol{\kappa}), \boldsymbol{\Phi})}{\|(0, \boldsymbol{\kappa})\|_{\boldsymbol{V}_p}} = \sup_{\boldsymbol{\kappa} \in \mathbf{M}} \frac{\langle \boldsymbol{\kappa}, \boldsymbol{\Phi} \rangle_{0,\Omega}}{\|\boldsymbol{\kappa}\|_{0,\Omega}} = \|\boldsymbol{\Phi}\|_{0,\Omega}.$$

(iii) By Poincaré–Friedrichs inequality, $\exists C(\Omega) > 0$ such that

$$|v|_{1,\Omega}^2 \geqslant C(\Omega) \|v\|_{1,\Omega}^2 \ \forall v \in W_0^{1,p}(\Omega) \hookrightarrow H_0^1(\Omega), \quad p > 2.$$

Hence,

$$\forall v \in W_0^{1,p}(\Omega) \hookrightarrow C^0(\bar{\Omega}), \quad p > 2,$$

$$\Phi^* = \left(-v\delta_{ij}\right), \quad 1 \leqslant i, j \leqslant 2 \;\Rightarrow\; \Phi^* \in \mathbf{W}, \quad M_{nt}(\Phi^*) = 0$$

$$\Rightarrow b^*(v, \Phi^*) = |v|_{1,\Omega}^2 \;\forall v \in W_0^{1,p}(\Omega), \quad p > 2$$

$$\Rightarrow \sup_{\Phi \in \mathbf{W}} \frac{b^*(v, \Phi)}{\|\Phi\|_{\mathbf{W}}} \geqslant \frac{b^*(v, \Phi^*)}{\|\Phi^*\|_{\mathbf{W}}} = \frac{|v|_{1,\Omega}^2}{\sqrt{2}\,\|v\|_{1,\Omega}} \geqslant \sigma_1 \|v\|_{1,\Omega}$$

with $\sigma_1 = C(\Omega)/\sqrt{2} > 0$. □

Now, we will prove the uniqueness of solution of the problem (Q)/(Q*).

**Theorem 3.4.** *The problem* (Q) (*consequently*, (Q*)) *has at most one solution.*

**Proof.** Let $(\boldsymbol{u}_1, \Psi_1)$, $(\boldsymbol{u}_2, \Psi_2) \in \boldsymbol{V}_p \times \mathbf{W}$ be any two solutions of (Q). Then, their difference $(\boldsymbol{u}^*, \Psi^*) = (\boldsymbol{u}_1 - \boldsymbol{u}_2, \Psi_1 - \Psi_2) \in \boldsymbol{V}_p \times \mathbf{W}$ satisfies:

$$A(\boldsymbol{u}^*, \boldsymbol{v}) + b(\boldsymbol{v}, \Psi^*) = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{V}_p,$$

$$b(\boldsymbol{u}^*, \Phi) = 0 \;\forall \Phi \in \mathbf{W} \Rightarrow \boldsymbol{u}^* \in \boldsymbol{Z} \Rightarrow b(\boldsymbol{u}^*, \Psi^*) = 0 \Rightarrow A(\boldsymbol{u}^*, \boldsymbol{u}^*) = 0$$

$$\Leftrightarrow \boldsymbol{u}^* = \mathbf{0} \quad \text{in } \boldsymbol{Z} \Rightarrow b(\boldsymbol{v}, \Psi^*) = 0 \;\forall \boldsymbol{v} \in \boldsymbol{V}_p$$

$$\Rightarrow \Psi^* = 0 \quad \text{in } \mathbf{W} \quad (\text{by virtue of } (3.27)). \qquad \square$$

The problem ($P_G$) and the problem (Q) (consequently, (Q*)) are related by:

**Theorem 3.5.** *If* $(\boldsymbol{u}, \Psi) \in \boldsymbol{V}_p \times \mathbf{W}$ *be the solution of* (Q) *with* $\boldsymbol{u} = (u, \boldsymbol{\kappa}^*)$, $\boldsymbol{\kappa}^* = (\kappa_{ij}^*)$, $1 \leqslant i, j \leqslant 2$, *then* $u \in H_0^2(\Omega)$ *and is the solution of* ($P_G$) *with* $\kappa_{ij}^* = u_{,ij}$, $1 \leqslant i, j \leqslant 2$. *Conversely, if the solution* $u \in H_0^2(\Omega)$ *of* ($P_G$) *belongs to* $H^3(\Omega) \cap H_0^2(\Omega)$ *and* $a_{ijkl}\, u_{,kl} \in H^1(\Omega)$ $\forall i, j = 1, 2$, *then the problem* (Q) *has a unique solution* $(\boldsymbol{u}, \Psi) \in \boldsymbol{V}_p \times \mathbf{W}$ *with* $\boldsymbol{u} = (u, \boldsymbol{\kappa}^*)$, $\boldsymbol{\kappa}^* = (\kappa_{ij}^*)$, $\kappa_{ij}^* = u_{,ij}$ $\forall i, j = 1, 2$, *and* $\Psi = (\psi_{ij})$, $\psi_{ij} = a_{ijkl}u_{,kl}$ $\forall i, j = 1, 2$, $\Psi$ *being the Lagrange multiplier of the method.*

**Proof.** Let $(\boldsymbol{u}, \Psi) \in \boldsymbol{V}_p \times \mathbf{W}$ be the solution of (Q). Then, $\boldsymbol{u} = (u, \boldsymbol{\kappa}^*) \in \boldsymbol{Z}$ and hence, $u \in H_0^2(\Omega)$, $\boldsymbol{\kappa}^* = (\kappa_{ij}^*)$, $\kappa_{ij}^* = u_{,ij}$ (Theorem 3.1). $\forall \boldsymbol{v} = (v, \boldsymbol{\kappa}) \in \boldsymbol{Z} \hookrightarrow \boldsymbol{V}_p$,

$$A(\boldsymbol{u}, \boldsymbol{v}) = F(\boldsymbol{v}) \quad \Rightarrow \quad \int_\Omega a_{ijkl} u_{,ij} v_{,kl} \,\mathrm{d}\Omega = \int_\Omega f v \,\mathrm{d}\Omega \quad \forall v \in H_0^2(\Omega),$$

since $\boldsymbol{v} = (v, \boldsymbol{\kappa}) \in \boldsymbol{Z} \Rightarrow v \in H_0^2(\Omega)$, $\boldsymbol{\kappa} = (\kappa_{ij})$, $\kappa_{ij} = v_{,ij}$. So, $u$ is the unique solution of ($P_G$) by Theorem 2.1.

Conversely, suppose that the solution $u \in H_0^2(\Omega)$ of ($P_G$) belongs to $H^3(\Omega) \cap H_0^2(\Omega)$. Set $\boldsymbol{u} = (u, \boldsymbol{\kappa}^*)$ with $\boldsymbol{\kappa}^* = (\kappa_{ij}^*)$, $\kappa_{ij}^* = u_{,ij}$ $\forall i, j = 1, 2$. Then $\boldsymbol{u} \in \boldsymbol{Z}$ by the Theorem 3.1. Hence, $b(\boldsymbol{u}, \Phi) = 0$ $\forall \Phi \in \mathbf{W}$, i.e. the equation (3.15) is satisfied. Thus, in order to prove that $(\boldsymbol{u}, \Psi) \in \boldsymbol{V}_p$

$\times\, \mathbf{W}$ with $\Psi=(\psi_{ij})$, $\psi_{ij}=a_{ijkl}u_{,kl}$ $\forall i, j=1, 2$, is the solution of (Q), it remains to show that $(\boldsymbol{u}, \Psi)$ satisfies the equation (3.14). In fact, $\forall \boldsymbol{v}=(v, \boldsymbol{\kappa})\in V_p$ with $\boldsymbol{\kappa}=(\kappa_{ij})$,

$$\begin{aligned} A(\boldsymbol{u}, \boldsymbol{v})+b(\boldsymbol{v}, \Psi) &= \int_\Omega a_{ijkl}\kappa^*_{ij}\kappa_{kl}\,\mathrm{d}\Omega \\ &\quad+\sum_{T\in T_h}\left(-\int_T(\kappa_{ij}\psi_{ij}+v_{,i}\psi_{ij,j})\,\mathrm{d}\Omega+\int_{\partial T}M_{nt}(\Psi)\frac{\partial v}{\partial t}\mathrm{d}s\right). \end{aligned}$$

Using Green's formula, $\forall \boldsymbol{v}\in H_0^2(\Omega)\times \mathbf{M}$,

$$\begin{aligned} A(\boldsymbol{u}, \boldsymbol{v})+b(\boldsymbol{v}, \Psi) &= \int_\Omega a_{ijkl}u_{,kl}\kappa_{ij}\,\mathrm{d}\Omega-\int_\Omega a_{ijkl}u_{,kl}\kappa_{ij}\,\mathrm{d}\Omega+\sum_{T\in T_h}\int_T v_{,ij}\psi_{ij}\,\mathrm{d}\Omega \\ &= \int_\Omega a_{ijkl}v_{,ij}u_{,kl}\,\mathrm{d}\Omega=a(u, v)=\int_\Omega fv\,\mathrm{d}\Omega=F(\boldsymbol{v}), \end{aligned} \tag{3.29}$$

since

$$\sum_{T\in T_h}\int_{\partial T}M_n(\Psi)\frac{\partial v}{\partial n}\mathrm{d}s=0 \quad\text{and}\quad u\in H_0^2(\Omega)\cap H^3(\Omega)$$

is the solution of ($P_G$). But $H_0^2(\Omega)\times\mathbf{M}$ is dense in $V_p$ by virtue of (1.3). Therefore, the equality (3.29) holds $\forall \boldsymbol{v}\in V_p$, and consequently, $(\boldsymbol{u}, \Psi)$ is the solution of (Q). □

**Remark 3.2.** The solution $(\boldsymbol{u}, \Psi)\in V_p\times\mathbf{W}$ of the mixed method scheme (Q) will be the saddle point of the functional $\mathscr{L}(\cdot, \cdot)$ defined by:

$$\mathscr{L}(\boldsymbol{v}, \Phi)=\tfrac12 A(\boldsymbol{v}, \boldsymbol{v})+b(\boldsymbol{v}, \Phi)-F(\boldsymbol{v}) \quad \forall \boldsymbol{v}\in V_p, \quad \forall \Phi\in\mathbf{W}$$

i.e.

$$\mathscr{L}(\boldsymbol{u}, \Phi)\leqslant\mathscr{L}(\boldsymbol{u}, \Psi)\leqslant\mathscr{L}(\boldsymbol{v}, \Psi) \quad \forall \boldsymbol{v}\in V_p, \quad \forall\Phi\in\mathbf{W},$$

since from (3.14)–(3.15),

$$\begin{aligned} \mathscr{L}(\boldsymbol{u}, \Psi) &= \tfrac12 A(\boldsymbol{u}, \boldsymbol{u})+b(\boldsymbol{u}, \Psi)-F(\boldsymbol{u})=\tfrac12 A(\boldsymbol{u}, \boldsymbol{u})-F(\boldsymbol{u}) \\ &= \tfrac12 A(\boldsymbol{u}, \boldsymbol{u})-A(\boldsymbol{u}, \boldsymbol{u})=-\tfrac12 A(\boldsymbol{u}, \boldsymbol{u}); \\ \mathscr{L}(\boldsymbol{u}, \Phi) &= \tfrac12 A(\boldsymbol{u}, \boldsymbol{u})+b(\boldsymbol{u}, \Phi)-F(\boldsymbol{u})=-\tfrac12 A(\boldsymbol{u}, \boldsymbol{u})=\mathscr{L}(\boldsymbol{u}, \Psi); \\ \mathscr{L}(\boldsymbol{v}, \Psi) &= \tfrac12 A(\boldsymbol{v}, \boldsymbol{v})+b(\boldsymbol{v}, \Psi)-F(\boldsymbol{v})=\tfrac12 A(\boldsymbol{v}, \boldsymbol{v})-A(\boldsymbol{u}, \boldsymbol{v}) \\ &= \tfrac12 A(\boldsymbol{v}, \boldsymbol{v})-A(\boldsymbol{u}, \boldsymbol{v})+\tfrac12 A(\boldsymbol{u}, \boldsymbol{u})-\tfrac12 A(\boldsymbol{u}, \boldsymbol{u}) \\ &= \tfrac12 A(\boldsymbol{u}-\boldsymbol{v}, \boldsymbol{u}-\boldsymbol{v})-\tfrac12 A(\boldsymbol{u}, \boldsymbol{u})\geqslant\mathscr{L}(\boldsymbol{u}, \Psi). \end{aligned}$$

**Example 3.1.** (The biharmonic problem (2.8)–(2.9)). Since $\psi_{ij}=a_{ijkl}u_{,kl}=u_{,ij}=\kappa^*_{ij}$, $\boldsymbol{\kappa}^*=\Psi$. Hence, we choose $\mathbf{M}=\mathbf{W}$, and the problem ($Q^*$) becomes: find $(u, \boldsymbol{\kappa}^*)\in W_0^{1,p}(\Omega)\times\mathbf{W}$, $(p>2)$, such that

$$\tilde a(\boldsymbol{\kappa}^*, \Phi)-b^*(u, \Phi)=0 \quad \forall\Phi\in\mathbf{W}, \tag{3.30}$$

$$b^*(v, \boldsymbol{\kappa}^*)=\langle f, v\rangle_{0,\Omega} \quad \forall v\in W_0^{1,p}(\Omega), \tag{3.31}$$

where $\tilde a(\boldsymbol{\kappa}^*, \Phi)=\int_\Omega\kappa^*_{ij}\phi_{ij}\,\mathrm{d}\Omega$, $b^*(\cdot, \cdot)$ is defined by (3.12) with $\mathbf{M}=\mathbf{W}$, which is the well known Hellan–Herrmann–Johnson mixed method scheme [10,13,14,16] for the biharmonic problem (2.7)–(2.9).

**Remark 3.3.** The mixed variational formulation (3.30)–(3.31) corresponds to a modified version of Hellinger–Reissner variational principle [20,23] which is a two-field principle.

**Remark 3.4.** The bilinear form $\tilde{a}(\cdot,\ \cdot)$ in (3.30)–(3.31) corresponds to that for the isotropic case of bending problem of elastic plate with $D=1$, $\nu=0$ in (3.36)–(3.37). Then $u=$ deflection, $\boldsymbol{\kappa}^* = \boldsymbol{\Psi}$, i.e. change in curvature tensor $\boldsymbol{\kappa}^* =$ bending moment tensor $\boldsymbol{\Psi}$. But for $D \neq 1$, $\nu \neq 0$, we have $\boldsymbol{\kappa}^* \neq \boldsymbol{\Psi}$.

**Example 3.2** (Bending problems of clamped elastic plates in (2.10)–(2.19)). Since $b(\cdot,\ \cdot)$ (res. $b^*(\cdot,\ \cdot)$) is defined by (3.18) (resp. (3.12)) for all the following cases we will show only the expression for $A(\cdot,\ \cdot)$ (resp. $\tilde{a}(\cdot,\ \cdot)$) in each case. The components of the solution vector $(u,\ \boldsymbol{\kappa}^*,\ \boldsymbol{\Psi})$ are deflection $u$, change in curvature tensor $\boldsymbol{\kappa}^* = (u_{,ij})$ and bending moment tensor $\boldsymbol{\Psi} = (\psi_{ij})$ with $\psi_{ij} = a_{ijkl}u_{,kl}$ with

$\psi_{ii} =$ bending moment in $x_i$-*direction* $\quad (i=1,\ 2)$,

$\psi_{12} = \psi_{21} =$ twisting moment in all the following cases:

$\forall \boldsymbol{v} = (v,\ \boldsymbol{\kappa}) \in \boldsymbol{V}_p,\quad \forall \boldsymbol{w} = (w,\ \boldsymbol{\eta}) \in \boldsymbol{V}_p$ with $\boldsymbol{\kappa} = (\kappa_{ij}) \in \mathbf{M}$, $\boldsymbol{\eta} = (\eta_{ij}) \in \mathbf{M}$,

(i) Anisotropic case (2.10)–(2.12):

$$\begin{aligned} A(\boldsymbol{v},\ \boldsymbol{w}) = \tilde{a}(\boldsymbol{\kappa},\ \boldsymbol{\eta}) = \int_\Omega \big[ & (D_{11}\kappa_{11} + 2D_{16}\kappa_{12} + D_{12}\kappa_{22})\eta_{11} \\ & + 2(D_{16}\kappa_{11} + 2D_{66}\kappa_{12} + D_{26}\kappa_{22})\eta_{12} \\ & + (D_{12}\kappa_{11} + 2D_{26}\kappa_{12} + D_{22}\kappa_{22})\eta_{22}\big]\ \mathrm{d}\Omega, \end{aligned} \tag{3.32}$$

$$\begin{aligned} &\psi_{11} = D_{11}u_{,11} + 2D_{16}u_{,12} + D_{12}u_{,22}; \qquad \psi_{22} = D_{12}u_{,11} + 2D_{26}u_{,12} + D_{22}u_{,22}; \\ &\psi_{12} = \psi_{21} = D_{16}u_{,11} + 2D_{66}u_{,12} + D_{26}u_{,22}; \end{aligned} \tag{3.33}$$

(ii) Orthotropic case (2.13)–(2.16):

$$\begin{aligned} A(\boldsymbol{v},\ \boldsymbol{w}) = \tilde{a}(\boldsymbol{\kappa},\ \boldsymbol{\eta}) = \int_D \big[ & (D_1\kappa_{11} + \nu_1 D_2\kappa_{22})\eta_{11} + 4D_t\kappa_{12}\eta_{12} \\ & + (\nu_2 D_1\kappa_{11} + D_2\kappa_{22})\eta_{22}\big]\ \mathrm{d}\Omega; \end{aligned} \tag{3.34}$$

$$\psi_{11} = D_1(u_{,11} + \nu_2 u_{,22}); \quad \psi_{22} = D_2(\nu_1 u_{,11} + u_{,22}); \qquad \psi_{12} = \psi_{21} = 2D_t u_{,12}; \tag{3.35}$$

(iii) Isotropic case (2.17)–(2.19):

$$A(\boldsymbol{v},\ \boldsymbol{w}) = \tilde{a}(\boldsymbol{\kappa},\ \boldsymbol{\eta}) = \int_\Omega D\big[(\kappa_{11} + \nu\kappa_{22})\eta_{11} + 2(1-\nu)\kappa_{12}\eta_{12} + (\nu\kappa_{11} + \kappa_{22})\eta_{22}\big]\ \mathrm{d}\Omega; \tag{3.36}$$

$$\psi_{11} = D(u_{,11} + \nu u_{,22}); \qquad \psi_{22} = D(\nu u_{,11} + u_{,22}); \qquad \psi_{12} = \psi_{21} = D(1-\nu)u_{,12}. \tag{3.37}$$

## 4. Mixed finite element approximation

Corresponding to the triangulation introduced earlier, we define finite-dimensional subspaces $X_{0h}$, $\mathbf{M}_h$, $\mathbf{W}_h$ of $W_0^{1,p}$ $(p>2)$, $\mathbf{M}$ and $\mathbf{W}$ respectively as follows: For positive integers $m \geqslant 1$,

$$X_h = \{ v_h : v_h \in C^0(\bar{\Omega}),\ v_h|_T \in P_m(T)\ \forall T \in T_h \}, \tag{4.1}$$

$$X_{0h} = \{ v_h : v_h \in X_h,\ v_h|_\Gamma = 0 \} \subset W_0^{1,p}(\Omega) \quad (p>2), \tag{4.2}$$

$$\mathbf{M}_h = \{ \kappa_h : \kappa_h = (\kappa_{hij}) \in \mathbf{M},\ \kappa_{hij}|_T \in P_{m-1}(T)\ \forall T \in T_h \} \subset \mathbf{M}, \tag{4.3}$$

$$\mathbf{W}_h = \{ \Phi_h : \Phi_h = (\phi_{hij}) \in \mathbf{W},\ \phi_{hij}|_T \in P_{m-1}(T)\ \forall T \in T_h \} \subset \mathbf{W}, \tag{4.4}$$

where $P_m(T)$ is the linear space of polynomials of degree $\leqslant m$ in $x_1$, $x_2$ on $T \in T_h$. Then

$$V_h = X_{0h} \times \mathbf{M}_h \subset V_p \quad (p>2). \tag{4.5}$$

Now, for the problem (Q*) we construct the corresponding mixed finite-element problem ($\mathrm{Q}_h^*$) under consideration as follows: Find $(u_h, \kappa_h^*, \Psi_h) \in X_{0h} \times \mathbf{M}_h \times \mathbf{W}_h$ such that

$$(\mathrm{Q}_h^*)\quad \begin{aligned} &\tilde{a}(\kappa_h^*, \eta_h) - [\Psi_h, \eta_h]_{0,\Omega} = 0 && \forall \eta_h = (\eta_{hij}) \in \mathbf{M}_h; && (4.6)\\ &b^*(u_h, \Phi_h) - [\kappa_h^*, \Phi_h]_{0,\Omega} = 0 && \forall \Phi_h = (\phi_{hij}) \in \mathbf{W}_h; && (4.7)\\ &b^*(v_h, \Psi_h) = \langle f, v_h \rangle_{0,\Omega} && \forall v_h \in X_{0h}; && (4.8) \end{aligned}$$

where the bilinear forms $\tilde{a}(\cdot,\cdot)$ and $b^*(\cdot,\cdot)$ are defined by (3.11) and (3.12) respectively.

The problem ($\mathrm{Q}_h^*$) can be rewritten in the form ($\mathrm{Q}_h$) which obviously corresponds to the problem (Q): Find $(\boldsymbol{u}_h, \Psi_h) \in V_h \times W_h$ such that

$$(\mathrm{Q}_h)\quad \begin{aligned} &A(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, \Psi_h) = F(\boldsymbol{v}_h) && \forall \boldsymbol{v}_h = (v_h, \eta_h) \in V_h, && (4.9)\\ &b(\boldsymbol{u}_h, \Phi_h) = 0 && \forall \Phi_h \in \mathbf{W}_h, && (4.10) \end{aligned}$$

where $\boldsymbol{u}_h = (u_h, \kappa_h^*) \in V_h$; $A(\cdot,\cdot)$, $b(\cdot,\cdot)$ and $F(\cdot)$ are defined by (3.16), (3.17)–(3.18) and (3.19) respectively.

**Remark 4.1.** By virtue of inclusions (4.2)–(4.5) the problem ($\mathrm{Q}_h^*$) (resp. ($\mathrm{Q}_h$)) is of conforming type [11].

Since both the mixed finite-element problems ($\mathrm{Q}_h^*$) and ($\mathrm{Q}_h$) are equivalent, we are free to choose anyone of them according to our convenience. Hence, for mathematical analysis we will choose ($\mathrm{Q}_h$) and for computational purpose we will recommend ($\mathrm{Q}_h^*$). Thus, for the proof of existence and uniqueness of solution of the mixed finite-element problem ($\mathrm{Q}_h$), we prepare similar results as in the case of the problem (Q).

Define the finite-dimensional subspace of $V_h$:

$$Z_h = \{ \boldsymbol{v}_h : \boldsymbol{v}_h = (v_h, \kappa_h) \in V_h,\ b(\boldsymbol{v}_h, \Phi_h) = 0\ \forall \Phi_h \in \mathbf{W}_h \}, \tag{4.11}$$

which is equipped with the norm topology of $V$. Since $Z_h \not\subset Z$ in general, we are to prove the $Z_h$-ellipticity of $A(\cdot,\cdot)$, for which we require the following assumption analogous to (3.28):

$$\text{(A3)}\quad \exists \sigma_2 > 0 \quad \text{such that} \quad \sup_{\Phi_h \in \mathbf{W}_h} \frac{b^*(v_h, \Phi_h)}{\| \Phi_h \|_{\mathbf{W}}} \geqslant \sigma_2 | v_h |_{1,\Omega} \quad \forall v_h \in X_{0h}.$$

**Theorem 4.1.** *If the assumption* (A3) *holds, then* $A(\cdot, \cdot)$ *is* $\boldsymbol{Z}_h$*-elliptic, i.e.*

$$\exists \sigma_0 > 0 \text{ such that } A(\boldsymbol{v}_h, \boldsymbol{v}_h) \geqslant \sigma_0 \| \boldsymbol{v}_h \|_V^2 \ \forall \boldsymbol{v}_h \in \boldsymbol{Z}_h. \tag{4.12}$$

**Proof.** Assume that (A3) holds. Let $\boldsymbol{v}_h = (v_h, \boldsymbol{\kappa}_h) \in \boldsymbol{Z}_h$. Then, $b(\boldsymbol{v}_h, \boldsymbol{\Phi}_h) = 0 \ \forall \boldsymbol{\Phi}_h \in \mathbf{W}_h$, and from (3.17) and (3.7),

$$b^*(v_h, \boldsymbol{\Phi}_h) = [\boldsymbol{\kappa}_h, \boldsymbol{\Phi}_h]_{0,\Omega} \ \forall \boldsymbol{\Phi}_h \in \mathbf{W}_h \Rightarrow \sup_{\boldsymbol{\Phi}_h \in \mathbf{W}_h} \frac{b^*(v_h, \boldsymbol{\Phi}_h)}{\| \boldsymbol{\Phi}_h \|_{\mathbf{W}}}$$

$$= \sup_{\boldsymbol{\Phi}_h \in \mathbf{W}_h} \frac{[\boldsymbol{\kappa}_h, \boldsymbol{\Phi}_h]_{0,\Omega}}{\| \boldsymbol{\Phi}_h \|_{\mathbf{W}}} \leqslant \sup_{\boldsymbol{\Phi}_h \in \mathbf{W}_h} \frac{[\boldsymbol{\kappa}_h, \boldsymbol{\Phi}_h]_{0,\Omega}}{\| \boldsymbol{\Phi}_h \|_{0,\Omega}} = \| \boldsymbol{\kappa}_h \|_{0,\Omega} \ \forall \boldsymbol{v}_h \in \boldsymbol{Z}_h. \tag{4.13}$$

Then, using (A3) and (4.13), we have

$$\sigma_2 | v_h |_{1,\Omega} \leqslant \| \boldsymbol{\kappa}_h \|_{0,\Omega} \ \forall \boldsymbol{v}_h = (v_h, \boldsymbol{\kappa}_h) \in \boldsymbol{Z}_h \Rightarrow \forall \boldsymbol{v}_h = (v_h, \boldsymbol{\kappa}_h) \in \boldsymbol{Z}_h,$$

$$\| \boldsymbol{v}_h \|_V^2 \leqslant C(\Omega) | v_h |_{1,\Omega}^2 + \| \boldsymbol{\kappa}_h \|_{0,\Omega}^2 \leqslant c_0 \| \boldsymbol{\kappa}_h \|_{0,\Omega}^2 \quad \text{with } c_0 = (1 + C(\Omega)/\sigma_2^2) > 0.$$

From the assumption (A2′), it follows that

$$\forall \boldsymbol{v}_h = (v_h, \boldsymbol{\kappa}_h) \in \boldsymbol{Z}_h,$$

$$A(\boldsymbol{v}_h, \boldsymbol{v}_h) \geqslant \alpha_0 \| \boldsymbol{\kappa}_h \|_{0,\Omega}^2 \geqslant \sigma_0 \| \boldsymbol{v}_h \|_V \quad \text{with } \sigma_0 = \alpha_0 / c_0 > 0. \qquad \square$$

**Remark 4.2.** The crucial assumption (A3) is the compatibility condition which the finite element spaces $X_{0h}$ and $\mathbf{W}_h$ defined in (4.2) and (4.4) respectively must satisfy in order that $A(\cdot, \cdot)$ is $\boldsymbol{Z}_h$-elliptic which will be essential for the proof of existence and uniqueness of solution of $(\mathrm{Q}_h)$, i.e. the spaces $X_{0h}$ and $\mathbf{W}_h$ in (4.2) and (4.4) respectively can *not* be chosen independently and in an arbitrary way.

**Lemma 4.1.**

$$\forall \boldsymbol{\Phi}_h \in \mathbf{W}_h, \quad \sup_{\boldsymbol{v}_h \in V_h} \frac{b(\boldsymbol{v}_h, \boldsymbol{\Phi}_h)}{\| \boldsymbol{v}_h \|_{V_p}} \geqslant \| \boldsymbol{\Phi}_h \|_{\mathbf{M}}. \tag{4.14}$$

**Proof.**

$$\sup_{\boldsymbol{v}_h = (v, \boldsymbol{\kappa}_h) \in V_h} \frac{b(\boldsymbol{v}_h, \boldsymbol{\Phi}_h)}{\| \boldsymbol{v}_h \|_{V_p}} \geqslant \sup_{(0, \boldsymbol{\kappa}_h) \in V_h} \frac{b((0, \boldsymbol{\kappa}_h), \boldsymbol{\Phi}_h)}{\| (0, \boldsymbol{\kappa}_h) \|_{V_p}}$$

$$= \sup_{\boldsymbol{\kappa}_h \in \mathbf{M}_h} \frac{[\boldsymbol{\kappa}_h, \boldsymbol{\Phi}_h]}{\| \boldsymbol{\kappa}_h \|_{0,\Omega}} = \| \boldsymbol{\Phi}_h \|_{0,\Omega} \quad \forall \boldsymbol{\Phi}_h \in \mathbf{W}_h.$$

Now, we can prove the existence and uniqueness of solution of $(\mathrm{Q}_h)/(\mathrm{Q}_h^*)$.

**Theorem 4.2.** *Under the assumption* (A3), *the mixed finite-element problem* $(\mathrm{Q}_h)$ (*consequently* $(\mathrm{Q}_h^*)$) *has a unique solution.*

**Proof.** Since $V_h$, $\mathbf{W}_h$ are finite-dimensional spaces, it is sufficient to show that if $(\boldsymbol{u}_h, \boldsymbol{\Psi}_h) \in V_h \times \mathbf{W}_h$ is a solution of the corresponding homogeneous problem:

$$\forall \boldsymbol{v}_h \in V_h, \quad A(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, \boldsymbol{\Psi}_h) = 0; \qquad \forall \boldsymbol{\Phi}_h \in \mathbf{W}_h, \quad b(\boldsymbol{u}_h, \boldsymbol{\Phi}_h) = 0,$$

then $\boldsymbol{u}_h = \mathbf{0}$, $\Psi_h = \mathbf{0}$. Then, $\boldsymbol{u}_h \in \boldsymbol{Z}_h$. Choosing $\boldsymbol{v}_h = \boldsymbol{u}_h$, we have $A(\boldsymbol{u}_h, \boldsymbol{u}_h) = 0 \Leftrightarrow \boldsymbol{u}_h = \mathbf{0}$ in $\boldsymbol{Z}_h$ (by (4.12)). Consequently, $b(\boldsymbol{v}_h, \Psi_h) = 0\ \forall \boldsymbol{v}_h \in \boldsymbol{V}_h \Rightarrow \Psi_h = \mathbf{0}$ by (4.14). □

The existence and uniqueness of solution of $(\mathrm{Q}_h)/(\mathrm{Q}_h^*)$ have been proved under the assumption (A3). Now, we will construct the spaces $\mathbf{W}_h$ and $X_{0h}$ in (4.4) and (4.2) respectively such that (A3) holds and also the space $\mathbf{M}_h$ in (4.3) with the help of finite elements [11] as follows: Let $\{L_{i,T}\}_{i=1}^3$ and $\{a_{i,T}\}_{i=1}^3$ denote three sides and three vertices of a given $T \in T_h$ with $\partial T = L_{1,T} \cup L_{2,T} \cup L_{3,T}$.

(I) For tensor valued functions $\Phi_h \in \mathbf{W}_h$ with $\Phi_h = (\phi_{hij})_{1 \leqslant i,j \leqslant 2}$, the set of degrees of freedom $(\Sigma_T^1)$ [10] in each triangle $T \in T_h$ may be chosen as the values of:

$$(\Sigma_T^1)\quad \begin{cases} \displaystyle\int_{L_{i,T}} M_n(\Phi_h) q \,\mathrm{d}s, \quad q \in P_{m-1}(L_{i,T}), \quad 1 \leqslant i \leqslant 3, \quad m \geqslant 1, & (4.15)\\ \displaystyle\int_T \phi_{hij} q \,\mathrm{d}T, \quad q \in P_{m-2}(T), \quad i,j = 1,2, \quad m \geqslant 2; & (4.16)\end{cases}$$

(II) For tensor valued functions $\boldsymbol{\kappa}_h \in \mathbf{M}_h$ with $\boldsymbol{\kappa}_h = (\kappa_{hij})_{1 \leqslant i,j \leqslant 2}$, the set of degrees of freedom $(\Sigma_T^2)$ in each $T \in T_h$ may be chosen *independently* as the values of:

$$(\Sigma_T^2)\quad \int_T \kappa_{hij} q \,\mathrm{d}T, \quad q \in P_{m-1}(T), \quad i,j = 1,2, \quad m \geqslant 1; \tag{4.17}$$

(III) For functions $v_h \in X_{0h}$, the set of degrees of freedom $(\Sigma_T^3)$ in each triangle $T \in T_h$ may be chosen as the values of:

$$(\Sigma_T^3)\quad \begin{cases} v_h(a_{i,T}), \quad 1 \leqslant i \leqslant 3, \quad m \geqslant 1, & (4.18)\\ \displaystyle\int_{L_{i,T}} v_h q \,\mathrm{d}s, \quad q \in P_{m-2}(L_{i,T}), \quad 1 \leqslant i \leqslant 3, \quad m \geqslant 2, & (4.19)\\ \displaystyle\int_T v_h q \,\mathrm{d}T, \quad q \in P_{m-3}(T), \quad m \geqslant 3. & (4.20)\end{cases}$$

Then,

$$\mathrm{Card}(\Sigma_T^1) = \dim W_T = 3m(m+1)/2, \qquad \mathrm{Card}(\Sigma_T^2) = \dim M_T = 3m(m+1)/2$$
$$\mathrm{Card}(\Sigma_T^3) = \dim X_T = (m+1)(m+2)/2;$$

where $W_T$ is the space of restrictions to $T$ of all $\Phi_h \in \mathbf{W}_h$, $M_T$ is the space of restrictions to $T$ of all $\boldsymbol{\kappa}_h \in \mathbf{M}_h$ and $X_T$ is the space of restrictions to $T$ of all $v_h \in X_{0h}$. Now, we introduce the following linear operators associated to these degrees of freedom (4.15)–(4.20), which will be quite helpful in the sequel, as follows:

Let $S_h$ (resp. $N_h$) denote the set of sides (resp. vertices) of triangles of $T_h$.

(I) $\Pi_h \in \mathscr{L}(\mathbf{W}; \mathbf{W}_h)$ is defined by [10]: $\forall \Phi \in \mathbf{W}$, $\Pi_h \Phi \in \mathbf{W}_h$ such that

$$\int_L M_n(\Phi - \Pi_h \Phi) q \,\mathrm{d}s = 0, \qquad q \in P_{m-1}(L), \quad L \in S_h, \quad m \geqslant 1, \tag{4.21}$$

$$\int_T \left(\phi_{hij} - (\Pi_h \Phi)_{ij}\right) q \,\mathrm{d}T = 0 \quad q \in P_{m-2}(T), \quad T \in T_h, \quad m \geqslant 2; \tag{4.22}$$

(II) $\underset{\sim}{\Pi} \in \mathscr{L}(\mathbf{M}; \mathbf{M}_h)$ is defined by $\forall \boldsymbol{\kappa} \in \mathbf{M}$, $\underset{\sim}{\Pi}_h \boldsymbol{\kappa} \in \mathbf{M}_h$ such that

$$\int_T (\kappa_{ij} - (\underset{\sim}{\Pi}_h \boldsymbol{\kappa})_{ij}) q \, \mathrm{d}T = 0, \quad q \in P_{m-1}(T), \quad T \in T_h, \quad m \geqslant 1; \tag{4.23}$$

(III) $\Pi_{0h} \in \mathscr{L}(W_0^{1,p}(\Omega); X_{0h})$, $p > 2$, is defined by: $\forall v \in W_0^{1,p}(\Omega) \hookrightarrow C^0(\bar{\Omega})$, $\Pi_{0h} v \in X_{0h}$ such that

$$(v - \Pi_{0h} v)(a) = 0 \qquad \forall a \in N_h, \tag{4.24}$$

$$\int_L (v - \Pi_{0h} v) q \, \mathrm{d}s = 0, \quad q \in P_{m-2}(L), \quad L \in S_h, \quad m \geqslant 2, \tag{4.25}$$

$$\int_T (v - \Pi_{0h} v) q \, \mathrm{d}T = 0, \quad q \in P_{m-3}(T), \quad T \in T_h, \quad m \geqslant 3. \tag{4.26}$$

Now, we define the linear operator $\Pi_h \in \mathscr{L}(V_p; V_h)$, $p > 2$, as follows: $\forall \boldsymbol{v} = (v, \boldsymbol{\kappa}) \in V_p$, $\Pi_h \boldsymbol{v} \in V_h = X_{0h} \times \mathbf{M}_h$ ((4.5)) such that

$$\Pi_h \boldsymbol{v} = (\Pi_{0h}, \underset{\sim}{\Pi}_h)(v, \boldsymbol{\kappa}) = (\Pi_{0h} v, \underset{\sim}{\Pi}_h \boldsymbol{\kappa}) \in X_{0h} \times \mathbf{M}_h = V_h, \tag{4.27}$$

where $\Pi_{0h} \in \mathscr{L}(W_0^{1,p}(\Omega); X_{0h})$, $p > 2$, $\underset{\sim}{\Pi}_h \in \mathscr{L}(\mathbf{M}; \mathbf{M}_h)$ are defined by (4.24)–(4.26) and (4.23) respectively.

Certain important properties associated with $\Pi_h \in \mathscr{L}(\mathbf{W}; \mathbf{W}_h)$ and $\Pi_h \in \mathscr{L}(V_p; V_h)$ will be proved now.

**Lemma 4.2.**

$$\forall \Phi \in \mathbf{W}, \quad b^*(v_h, \Phi - \Pi_h \Phi) = 0 \quad \forall v_h \in X_{0h}, \tag{4.28}$$

*where* $b^*(\cdot, \cdot)$ *is defined by* (3.12), $\Pi_h \in \mathscr{L}(\mathbf{W}; \mathbf{W}_h)$ *is defined by* (4.21)–(4.22).

**Proof.** From (3.12), we have: $\forall \Phi \in \mathbf{W}$, for $\partial T = \cup_{i=1}^3 L_{i,T}$,

$$\begin{aligned} & b^*(v_h, \Phi - \Pi_h \Phi) \\ & = \sum_{T \in T_h} \left[ -\int_T ((\Phi - \Pi_h \Phi)_{ij})_{,j} v_{h,i} \, \mathrm{d}T + \sum_{i=1}^{3} \int_{L_{i,T}} M_{nt}(\Phi - \Pi_h \Phi) \frac{\partial v_h}{\partial t} \mathrm{d}s \right] \\ & = \sum_{T \in T_h} \left[ \int_T (\Phi - \Pi_h \Phi)_{ij} v_{h,ij} \, \mathrm{d}T - \sum_{i=1}^{3} \int_{L_{i,T}} M_n(\Phi - \Pi_h \Phi) \frac{\partial v_h}{\partial n} \mathrm{d}s \right], \end{aligned} \tag{4.29}$$

which follows from the application of Green's formula

$$\begin{aligned} -\int_T (\Phi - \Pi_h \Phi)_{ij,j} v_{h,i} \, \mathrm{d}T = & \int_T (\Phi - \Pi_h \Phi)_{ij} v_{h,ij} \, \mathrm{d}T \\ & - \int_{\partial T} \left[ M_n(\Phi - \Pi_h \Phi) \frac{\partial v_h}{\partial n} + M_{nt}(\Phi - \Pi_h \Phi) \frac{\partial v_h}{\partial t} \right] \mathrm{d}s. \end{aligned}$$

Since

$$v_{h,ij}|_T \in P_{m-2}(T), \quad \left. \frac{\partial v_h}{\partial n} \right|_{L_{i,T}} \in P_{m-1}(L_{i,T}), \qquad 1 \leqslant i \leqslant 3, \quad \forall T \in T_h,$$

each of the two integrals in the square bracket in (4.29) vanishes $\forall T \in T_h$ by the definition of $\Pi_h \in \mathscr{L}(\mathbf{W}; \mathbf{W}_h)$ in (4.21)–(4.22). □

**Lemma 4.3.**

$$\forall \boldsymbol{v} \in V_p, \quad p > 2, \quad b(\boldsymbol{v} - \Pi_h \boldsymbol{v}, \Phi_h) = 0 \ \forall \Phi_h \in \mathbf{W}_h, \tag{4.30}$$

*where* $\Pi_h \in \mathscr{L}(V_p; V_h)$ *is defined by* (4.27).

**Proof.** $\forall \boldsymbol{v} \in V_p$, $\forall \Phi_h \in \mathbf{W}_h$,

$$\begin{aligned} b(\boldsymbol{v} - \Pi_h \boldsymbol{v}, \Phi_h) &= \sum_{T \in T_h} \left( \int_T - \left[ \left( \kappa_{ij} - (\underset{\sim}{\Pi}_h \kappa)_{ij} \right) \phi_{hij} + (v - \Pi_{0h} v)_{,i} \phi_{hij,j} \right] \mathrm{d}\Omega \right. \\ &\qquad \left. + \int_{\partial T} M_{nt}(\Phi_h) \frac{\partial}{\partial t}(v - \Pi_{0h} v) \, \mathrm{d}s \right) \\ &= \sum_{T \in T_h} \left[ \int_T (v - \Pi_{0h} v) \phi_{hij,ij} \, \mathrm{d}\Omega \right. \\ &\qquad \left. + \int_{\partial T} \left\{ M_{nt}(\Phi_h) \frac{\partial}{\partial t}(v - \Pi_{0h} v) - Q_n(\Phi_h)(v - \Pi_{0h} v) \right\} \mathrm{d}s \right], \end{aligned}$$

which follows from (4.23) and (3.1).

Now, integrating by parts the term

$$\int_{\partial T} M_{nt}(\Phi_h) \frac{\partial}{\partial t}(v - \Pi_{0h} v) \, \mathrm{d}s$$

along each side $L_{i,T}$, $1 \leqslant i \leqslant 3$, of a given triangle $T \in T_h$, we get

$$\begin{aligned} \int_{\partial T} M_{nt}(\Phi_h) \frac{\partial (v - \Pi_{0h} v)}{\partial t} \mathrm{d}s &= - \sum_{i=1}^{3} \int_{L_{i,T}} \frac{\partial}{\partial t}(M_{nt}(\Phi_h))(v - \Pi_{0h} v) \, \mathrm{d}s \\ &\quad + \sum_{i=1}^{3} J_{i,T}(\Phi_h)(v - \Pi_{0h} v)(a_{i,T}), \end{aligned}$$

where $J_{i,T}(\Phi_h)$ is a convenient expression depending on $\Phi_h(a_{i,T})$. Hence, from (4.30) and (4.26), $\forall \boldsymbol{v} \in V_p$, $\forall \Phi_h \in \mathbf{W}_h$,

$$\begin{aligned} b(\boldsymbol{v} - \Pi_h \boldsymbol{v}, \Phi_h) &= \sum_{T \in T_h} \left[ - \sum_{i=1}^{3} \int_{L_{i,T}} \left( \frac{\partial}{\partial t} M_{nt}(\Phi_h) + Q_n(\Phi_h) \right) (v - \Pi_{0h} v) \, \mathrm{d}s \right. \\ &\qquad \left. + \sum_{i=1}^{3} J_{i,T}(\Phi_h)(v - \Pi_{0h} v)(a_{i,T}) \right] \\ &= \sum_{L \in S_h} \int_L q(L, \Phi_h)(v - \Pi_{0h} v) \, \mathrm{d}s + \sum_{a \in N_h} B(a, \Phi_h)(v - \Pi_{0h} v)(a), \end{aligned}$$

where $q(L, \Phi_h)$ is a polynomial of degree $\leqslant m - 2$ in the variable $s$. Then, the result follows from the definition of the operator $\Pi_{0h}$ in (4.24)–(4.26).

**Lemma 4.4.** *Let* $\{T_h\}$ *be a family of regular triangulations of* $\bar{\Omega}$. *Let* $\Pi_h \in \mathscr{L}(\mathbf{W}; \mathbf{W}_h)$, $\underset{\sim}{\Pi}_h \in \mathscr{L}(\mathbf{M}; \mathbf{M}_h)$ *and* $\Pi_{0h} \in \mathscr{L}(W_0^{1,p}(\Omega), X_{0h})$, $p > 2$, *be the linear operators defined in* (4.21)–(4.22), (4.23) *and* (4.24)–(4.26) *respectively. Then, the following well known estimates* [10], [11] *hold*: $\exists c_i \geqslant 0$, $1 \leqslant i \leqslant 6$, *independent of* $h$,

$$\text{(I)} \qquad \forall \Phi \in \mathbf{W}, \quad \| \Pi_h \Phi \|_{\mathbf{W}} \leqslant C_1 \| \Phi \|_{\mathbf{W}}, \tag{4.31}$$

$$\text{(II)} \qquad \forall \Phi \in \mathbf{W} \cap (H^m(\Omega))^4, \quad \| \Phi - \Pi_h \Phi \|_{0,\Omega} \leqslant C_2 h^m | \Phi |_{m,\Omega}, \tag{4.32}$$

$$\text{(II)} \qquad \forall \kappa \in \mathbf{M}, \quad \| \underset{\sim}{\Pi}_h \kappa \|_{\mathbf{M}} \leqslant C_3 \| \kappa \|_{\mathbf{M}}, \tag{4.33}$$

$$\forall \kappa \in \mathbf{M} \cap (H^m(\Omega))^4, \quad \| \kappa - \underset{\sim}{\Pi}_h \kappa \|_{0,\Omega} \leqslant C_4 h^m | \kappa |_{m,\Omega}, \tag{4.34}$$

$$\text{(III)} \qquad \forall v \in W_0^{1,p}(\Omega), \quad p > 2, \quad \| \Pi_{0h} v \|_{1,\Omega} \leqslant C_5 \| v \|_{1,\Omega}, \tag{4.35}$$

$$\forall v \in W_0^{1,p}(\Omega) \cap H^{m+1}(\Omega), \quad \| v - \Pi_{0h} v \|_{1,\Omega} \leqslant C_6 h^m | v |_{m+1,\Omega}. \tag{4.36}$$

Now, we will prove that (A3) holds for $\mathbf{W}_h$ and $X_{0h}$:

**Lemma 4.5.** *Let* $\{T_h\}$, $0 < h < 1$, *be a family of regular triangulations of* $\bar{\Omega}$. *Let* $\mathbf{W}_h$ *and* $X_{0h}$ *be constructed with the help of finite element degrees of freedom* $(\Sigma_T^1)$ *and* $(\Sigma_T^3)$ *in* (4.15)–(4.16) *and* (4.18)–(4.20) *respectively. Then the assumption* (A3) *holds.*

**Proof.** Let $\Pi_h \in \mathscr{L}(\mathbf{W}; \mathbf{W}_h)$ be defined by (4.21)–(4.22). Then

$$\begin{aligned} \sup_{\Phi_h \in \mathbf{W}_h} \frac{b^*(v_h, \Phi_h)}{\| \Phi_h \|_{\mathbf{W}}} &\geqslant \sup_{\Phi \in \mathbf{W}} \frac{b^*(v_h, \Pi_h \Phi)}{\| \Pi_h \Phi \|_{\mathbf{W}}} = \sup_{\Phi \in \mathbf{W}} \frac{b^*(v_h, \Pi_h \Phi)}{\| \Phi \|_{\mathbf{W}}} \frac{\| \Phi \|_{\mathbf{W}}}{\| \Pi_h \Phi \|_{\mathbf{W}}} \\ &\geqslant \frac{1}{C_1} \sup_{\Phi \in \mathbf{W}} \frac{b^*(v_h, \Phi)}{\| \Phi \|_{\mathbf{W}}} \quad \text{(by (4.31) and (4.28))}. \end{aligned}$$

Then, from (3.28) we have

$$\sup_{\Phi_h \in \mathbf{W}_h} \frac{b^*(v_h, \Phi_h)}{\| \Phi_h \|_{\mathbf{W}}} \geqslant \sigma_2 | v_h |_{1,\Omega} \quad \forall v_h \in X_{0h} \quad \text{with } \sigma_2 = \frac{\sigma_1}{C_1},$$

i.e. the assumption (A3) holds.

## 5. Reduction of $(\mathrm{Q}_h^*)$ to matrix equations

For computational purpose any one of the two equivalent mixed finite-element problems $(\mathrm{Q}_h^*)$ and $(\mathrm{Q}_h)$ is to be reduced to matrix equations. In order to develop an outline of this reduction procedure, we will consider $(\mathrm{Q}_h^*)$ in (4.6)–(4.8) instead of $(\mathrm{Q}_h)$ for the sake of convenience.

Let $\{\mu_h^i\}_{i=1}^{L1}$, $\{\Phi_h^j\}_{j=1}^{M1}$ and $\{\chi_h^k\}_{k=1}^{N1}$ be bases in $\mathbf{M}_h$, $\mathbf{W}_h$ and $X_{0h}$ having dimensions $L1$, $M1$, $N1$ respectively, such that (A3) holds. Then, the components of the solution vector $(u_h, \kappa_h^*, \Psi_h) \in X_{0h} \times \mathbf{M}_h \times \mathbf{W}_h$ can be represented as follows:

$$\kappa_h^* = \sum_{i=1}^{L1} \alpha_i \mu_h^i, \qquad \Psi_h = \sum_{j=1}^{M1} \beta_j \Phi_h^j, \qquad u_h = \sum_{k=1}^{N1} \gamma_k \chi_h^k, \tag{5.1}$$

where $\alpha_i$, $\beta_j$, $\gamma_k \in \mathbb{R}$ for $1 \leqslant i \leqslant L1$, $1 \leqslant j \leqslant M1$, $1 \leqslant k \leqslant N1$. Then from (4.6)–(4.8), we have:

$$\tilde{a}\left(\sum_{i=1}^{L1} \alpha_i \mu_h^i,\ \mu_h^j\right) - \left[\sum_{k=1}^{M1} \beta_k \Phi_h^k,\ \mu_h^j\right]_{0,\Omega} = 0, \quad 1 \leqslant j \leqslant L1;$$

$$b^*\left(\sum_{k=1}^{N1} \gamma_k \chi_h^k,\ \Phi_h^j\right) - \left[\sum_{i=1}^{L1} \alpha_i \mu_h^i,\ \Phi_h^j\right]_{0,\Omega} = 0, \quad 1 \leqslant j \leqslant M1;$$

$$b^*\left(\chi_h^k,\ \sum_{j=1}^{M1} \beta_j \Phi_h^j\right) = \langle f,\ \chi_h^k \rangle_{0,\Omega}, \quad 1 \leqslant k \leqslant N1$$

$$\Rightarrow \sum_{i=1}^{L1} \alpha_i \tilde{a}\left(\mu_h^i,\ \mu_h^j\right) - \sum_{k=1}^{M1} \beta_k \left[\Phi_h^k,\ \mu_h^j\right]_{0,\Omega} = 0, \quad 1 \leqslant j \leqslant L1,$$

$$-\sum_{k=1}^{N1} \gamma_k b^*\left(\chi_h^k,\ \Phi_h^j\right) + \sum_{i=1}^{L1} \alpha_i \left[\mu_h^i,\ \Phi_h^j\right]_{0,\Omega} = 0, \quad 1 \leqslant j \leqslant M1,$$

$$\sum_{j=1}^{M1} \beta_j b^*\left(\chi_h^k,\ \Phi_h^j\right) = \langle f,\ \chi_h^k \rangle_{0,\Omega}, \quad 1 \leqslant k \leqslant N1.$$

Now, the problem $(Q_h^*)$ can be written in the matrix equation form as follows: Find $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}) \in \mathbb{R}^{L1+M1+N1}$ such that

$$[A]\boldsymbol{\alpha} - [C]\boldsymbol{\beta} = \mathbf{0}, \tag{5.2}$$

$$-[C]^{\mathrm{T}}\boldsymbol{\alpha} + [B]\boldsymbol{\gamma} = \mathbf{0}, \tag{5.3}$$

$$[B]^{\mathrm{T}}\boldsymbol{\beta} = \mathbf{F}, \tag{5.4}$$

where

$$[A] = \left(A_{ij}\right)_{1 \leqslant i,j \leqslant L1} \quad \text{with } A_{ij} = \tilde{a}\left(\mu_h^j,\ \mu_h^i\right) = A_{ji} \tag{5.5}$$

is symmetric, positive-definite by virtue of the $\boldsymbol{Z}_h$ ellipticity of $A(\cdot,\ \cdot)$

$$[C] = \left(C_{ij}\right)_{\substack{1 \leqslant i \leqslant L1 \\ 1 \leqslant j \leqslant M1}} \quad \text{with } C_{ij} = \left[\Phi_h^j,\ \mu_h^i\right]_{0,\Omega}, \tag{5.6}$$

$$[B] = \left(B_{ij}\right)_{\substack{1 \leqslant i \leqslant M1 \\ 1 \leqslant j \leqslant N1}} \quad \text{with } B_{ij} = b^*\left(\chi_h^j,\ \Phi_h^i\right), \tag{5.7}$$

$[C]^{\mathrm{T}} = \left(C_{ji}\right)_{\substack{1 \leqslant j \leqslant M1 \\ 1 \leqslant i \leqslant L1}}$ and $[B]^{\mathrm{T}} = \left(B_{ji}\right)_{\substack{1 \leqslant j \leqslant N1 \\ 1 \leqslant i \leqslant M1}}$ are transposes of $[C]$ and $[B]$ respectively,

$$\mathbf{F} = \left(F_1,\ F_2, \ldots,\ F_k, \ldots,\ F_{N1}\right)^{\mathrm{T}} \quad \text{with } F_k = \langle f,\ \chi_h^k \rangle_{0,\Omega}, \tag{5.8}$$

$$\boldsymbol{\alpha} = \left(\alpha_1,\ \alpha_2, \ldots,\ \alpha_{L1}\right)^{\mathrm{T}}; \qquad \boldsymbol{\beta} = \left(\beta_1,\ \beta_2, \ldots,\ \beta_{M1}\right)^{\mathrm{T}}, \qquad \boldsymbol{\gamma} = \left(\gamma_1,\ \gamma_2, \ldots,\ \gamma_{N1}\right)^{\mathrm{T}}. \tag{5.9}$$

Since basis functions in the basis $\{\mu_h^i\}_{i=1}^{L1}$ of $\mathbf{M}_h$ (4.3) can be made discontinuous at the interelement boundaries of $T_h$, degrees of freedom for the unknown change in curvature tensor field $\kappa_h^*$ can be assumed *independently* in each triangle $T \in T_h$ such that $[A]$ has the simple block diagonal structure, i.e.

$$[A] = \begin{bmatrix} [A_{T_1}] & & & \mathbf{0} \\ & [A_{T_2}] & & \\ & & \ddots & \\ \mathbf{0} & & & [A_{T_{NT}}] \end{bmatrix} \tag{5.10}$$

with element stiffness matrices $[A_{T_k}]$, $1 \leqslant k \leqslant NT$, $[A_{T_k}]$ being the contribution to $[A]$ from the $k$th triangle, and $NT$ being the total number of triangles. Then,

$$[A]^{-1} = \left[ [A_{T_1}]^{-1} [A_{T_2}]^{-1} \cdots [A_{T_{NT}}]^{-1} \right], \tag{5.11}$$

where only diagonal entries are shown, is obtained by inverting the element stiffness matrices $[A_{T_k}]$, $1 \leqslant k \leqslant NT$, immediately after their construction at the element level, $[A_{T_k}]$'s being of very small size in comparison with $[A]$ (in actual computation, the inversion operation is to be understood in the sense of solving relevant matrix equation(s)).

Then, from (5.2), we have

$$\boldsymbol{\alpha} = [A]^{-1}[C]\boldsymbol{\beta}, \tag{5.12}$$

where $[A]^{-1}$ is defined by (5.11). Now, eliminating $\boldsymbol{\alpha}$ from (5.3)–(5.4), we have

$$-[C]^{\mathrm{T}}[A]^{-1}[C]\boldsymbol{\beta} + [B]\boldsymbol{\gamma} = \mathbf{0}, \tag{5.13}$$

$$[B]^{\mathrm{T}}\boldsymbol{\beta} = \mathbf{F}, \tag{5.14}$$

or

$$\begin{bmatrix} [\mathbb{A}] & [\mathbb{B}] \\ [\mathbb{B}]^{\mathrm{T}} & [0] \end{bmatrix} \begin{bmatrix} \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbb{F} \end{bmatrix}, \tag{5.15}$$

where $[\mathbb{A}] = [C]^{\mathrm{T}}[A]^{-1}[C]$ is symmetric, positive-definite by virtue of positive-definiteness of $[A]^{-1}$;

$$[\mathbb{B}] = -[B], \qquad [\mathbb{B}]^{\mathrm{T}} = -[B]^{\mathrm{T}}, \qquad \mathbb{F} = -\mathbf{F}, \tag{5.16}$$

$[B]$ and $\mathbf{F}$ are defined by (5.7) and (5.8) respectively. Solving (5.13)–(5.14), $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are determined and then using (5.12), $\boldsymbol{\alpha}$ is found. Finally, $u_h$, $\kappa_h^*$, $\Psi_h$ are determined from (5.1).

## 6. Error estimates

Let $\{T_h\}$, $0 < h < 1$, be a family of regular triangulations of $\overline{\Omega}$. Corresponding to each $T_h$, let $\mathbf{W}_h$, $\mathbf{M}_h$ and $X_{0h}$ defined in (4.4), (4.3) and (4.2) be constructed with the help of degrees of freedom $(\Sigma_T^1)$, $(\Sigma_T^2)$ and $(\Sigma_T^3)$ in (4.15)–(4.16), (4.17), (4.18)–(4.20) respectively, for which all the estimates (4.31)–(4.36) in Lemma (4.4) and the assumption (A3) hold.

**Theorem 6.1.** *Let* $(\boldsymbol{u}, \Psi) = ((u, \kappa^*), \Psi) \in V_p \times \mathbf{W}$, $(p > 2)$ *and* $(\boldsymbol{u}_h, \Psi_h) = ((u_h, \kappa_h^*), \Psi_h) \in V_h \times \mathbf{W}_h$ *be the solutions of* (Q) *and* $(\mathrm{Q}_h)$ *respectively. Then, the following estimates hold*: $\exists \bar{C}_1, \bar{C}_2 > 0$, *independent of* $h$, *such that*

$$\| \boldsymbol{u} - \boldsymbol{u}_h \|_V \leqslant \bar{C}_1 ( \| \boldsymbol{u} - \Pi_h \boldsymbol{u} \|_V + \| \Psi - \Pi_h \Psi \|_{0,\Omega}), \tag{6.1}$$

$$\| \Psi - \Psi_h \|_{0,\Omega} \leqslant \bar{C}_2 ( \| \boldsymbol{u} - \boldsymbol{u}_h \|_V + \| \Psi - \Pi_h \Psi \|_{0,\Omega}), \tag{6.2}$$

*where* $\| \cdot \|_V$ *is defined by* (3.4), $\Pi_h \in \mathscr{L}(\mathbf{W}; \mathbf{W}_h)$ *and* $\Pi_h \in \mathscr{L}(V_p; V_h)$ *are defined by* (4.21)–(4.22) *and* (4.27) *respectively.*

**Proof.**

$$\forall \boldsymbol{v}_h \in \boldsymbol{Z}_h, \quad \| \boldsymbol{u} - \boldsymbol{u}_h \|_V \leqslant \| \boldsymbol{u} - \boldsymbol{v}_h \|_V + \| \boldsymbol{u}_h - \boldsymbol{v}_h \|_V, \quad \boldsymbol{u}_h \in \boldsymbol{Z}_h \tag{6.3}$$

$$\begin{aligned}
&\forall \boldsymbol{v}_h \in \boldsymbol{Z}_h, \quad \forall \Phi_h \in \mathbf{W}_h, \quad b(\boldsymbol{u}_h - \boldsymbol{v}_h, \Phi_h) = 0 \\
&\quad \Rightarrow \forall \boldsymbol{v}_h \in \boldsymbol{Z}_h, \quad \forall \Phi_h \in \mathbf{W}_h, \\
&\quad\quad b(\boldsymbol{u}_h - \boldsymbol{v}_h, \Psi - \Psi_h) = b(\boldsymbol{u}_h - \boldsymbol{v}_h, \Psi - \Phi_h) = b(\boldsymbol{u}_h - \boldsymbol{v}_h, \Psi - \Pi_h \Psi) \\
&\quad \Rightarrow \| \boldsymbol{u}_h - \boldsymbol{v}_h \|_V \frac{b(\boldsymbol{u}_h - \boldsymbol{v}_h, \Psi - \Psi_h)}{\| \boldsymbol{u}_h - \boldsymbol{v}_h \|_V} \leqslant \| \boldsymbol{u}_h - \boldsymbol{v}_h \|_V \sup_{\boldsymbol{w}_h \in V_h} \frac{b(\boldsymbol{w}_h, \Psi - \Pi_h \Psi)}{\| \boldsymbol{w}_h \|_V} \quad \forall \boldsymbol{v}_h \in \boldsymbol{Z}_h, \\
&\quad \boldsymbol{v}_h \neq \boldsymbol{u}_h.
\end{aligned} \tag{6.4}$$

But from (4.12), $\forall \boldsymbol{v}_h \in \boldsymbol{Z}_h$,

$$\begin{aligned}
\sigma_0 \| \boldsymbol{u}_h - \boldsymbol{v}_h \|_V^2 &\leqslant A(\boldsymbol{u}_h - \boldsymbol{v}_h, \boldsymbol{u}_h - \boldsymbol{v}_h) \\
&= A(\boldsymbol{u} - \boldsymbol{v}_h, \boldsymbol{u}_h - \boldsymbol{v}_h) + A(\boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{v}_h) \\
&= A(\boldsymbol{u} - \boldsymbol{v}_h, \boldsymbol{u}_h - \boldsymbol{v}_h) + b(\boldsymbol{u}_h - \boldsymbol{v}_h, \Psi - \Psi_h) \\
&\leqslant \| A \| \, \| \boldsymbol{u} - \boldsymbol{v}_h \|_V \| \boldsymbol{u}_h - \boldsymbol{v}_h \| \boldsymbol{v} + \| \boldsymbol{u}_h - \boldsymbol{v}_h \|_V \sup_{\boldsymbol{w}_h \in V_h} \frac{b(\boldsymbol{w}_h, \Psi - \Pi_h \Psi)}{\| \boldsymbol{w}_h \|_V}
\end{aligned}$$

(by virtue of the continuity of $A(\cdot, \cdot)$)

$$\Rightarrow \sigma_0 \inf_{\boldsymbol{v}_h \in \boldsymbol{Z}_h} \| \boldsymbol{u}_h - \boldsymbol{v}_h \|_V \leqslant \| A \| \inf_{\boldsymbol{v}_h \in \boldsymbol{Z}_h} \| \boldsymbol{u} - \boldsymbol{v}_h \|_V + \sup_{\boldsymbol{w}_h \in V_h} \frac{b(\boldsymbol{w}_h, \Psi - \Pi_h \Psi)}{\| \boldsymbol{w}_h \|_V}. \tag{6.5}$$

But from Lemma (4.3), $b(\boldsymbol{u} - \Pi_h \boldsymbol{u}, \Phi_h) = 0$ and $b(\boldsymbol{u}, \Phi_h) = 0 \ \forall \Phi_h \in \mathbf{W}_h$ (from (4.10))

$$\begin{aligned}
&\Rightarrow b(\Pi_h \boldsymbol{u}, \Phi_h) = 0 \ \forall \Phi_h \in \mathbf{W}_h, \quad \Pi_h \boldsymbol{u} \in V_h \Rightarrow \Pi_h \boldsymbol{u} \in \boldsymbol{Z}_h, \\
&\Rightarrow \inf_{\boldsymbol{v}_h \in z_h} \| \boldsymbol{u} - \boldsymbol{v}_h \|_V \leqslant \| \boldsymbol{u} - \Pi_h \boldsymbol{u} \|_V.
\end{aligned} \tag{6.6}$$

Using Green's formula, we get $\forall \boldsymbol{v}_h = (v_h, \kappa_h) \in V_h$,

$$\begin{aligned}
b(\boldsymbol{v}_h, \Psi - \Pi_h \Psi) = &- [\kappa_h, \Psi - \Pi_h \Psi]_{0,\Omega} + \sum_{T \in T_h} \int_T \left( v_{h,ij} \left( \psi_{ij} - (\Pi_h \Psi)_{ij} \right) \right) \mathrm{d}\Omega \\
&- \sum_{L \in S_h} \int_L M_n (\Psi - \Pi_h \Psi) \frac{\partial v_h}{\partial n} \, \mathrm{d}s = -[\kappa_h, \Psi - \Pi_h \Psi]_{0,\Omega},
\end{aligned}$$

since $v_{h,ij}|_T \in P_{m-2}$ and $(\partial v_h/\partial n)|_{\partial T}$ is a polynomial of degree $\leqslant m-1$ in the variable $s$, both the integrals vanish $\forall T \in T_h$, $\forall L \in S_h$ by virtue of the definition of the operator $\Pi_h$. Then, $\forall \boldsymbol{v}_h = (v_h, \boldsymbol{\kappa}_h) \in \boldsymbol{V}_h$,

$$\frac{|b(\boldsymbol{v}_h, \Psi - \Pi_h\Psi)|}{\|\boldsymbol{v}_h\|_V} \leqslant \frac{|b(\boldsymbol{v}_h, \Psi - \Pi_h\Psi)|}{\|\boldsymbol{\kappa}_h\|_{0,\Omega}} = \frac{|[\boldsymbol{\kappa}_h, \Psi - \Pi_h\Psi]_{0,\Omega}|}{\|\boldsymbol{\kappa}_h\|_{0,\Omega}}$$

$$\Rightarrow \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{|b(\boldsymbol{v}_h, \Psi - \Pi_h\Psi)|}{\|\boldsymbol{v}_h\|_V} \leqslant \|\Psi - \Pi_h\Psi\|_{0,\Omega} \tag{6.7}$$

(applying Cauchy–Schwarz inequality). Since

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_V \leqslant \inf_{\boldsymbol{v}_h \in \boldsymbol{Z}_h} \|\boldsymbol{u} - \boldsymbol{v}_h\| + \inf_{\boldsymbol{v}_h \in \boldsymbol{Z}_h} \|\boldsymbol{u}_h - \boldsymbol{v}_h\|,$$

combining (6.5)–(6.7), the result (6.1) is obtained with $\bar{c}_1 = \max\{\|A\|/\sigma_0, 1/\sigma_0\}$.

Now, we will prove (6.2).

$$\forall \Phi_h \in \mathbf{W}_h, \quad \|\Psi - \Psi_h\|_{0,\Omega} \leqslant \|\Psi - \Phi_h\|_{0,\Omega} + \|\Psi_h - \Phi_h\|_{0,\Omega}. \tag{6.8}$$

From (4.14), $\forall \Phi_h \in \mathbf{W}_h$,

$$\begin{aligned}\|\Psi_h - \Phi_h\|_{0,\Omega} &\leqslant \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{b(\boldsymbol{v}_h, \Psi_h - \Phi_h)}{\|\boldsymbol{v}_h\|_{V_p}} \\ &\leqslant \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{b(\boldsymbol{v}_h, \Psi - \Phi_n)}{\|\boldsymbol{v}_h\|_{V_p}} + \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{b(\boldsymbol{v}_h, \Psi_h - \Psi)}{\|\boldsymbol{v}_h\|_{V_p}}.\end{aligned} \tag{6.9}$$

Since $(\boldsymbol{u}, \Psi) \in \boldsymbol{V}_p \times \mathbf{W}$, $(\boldsymbol{u}_h, \Psi_h) \in \boldsymbol{V}_h \times \mathbf{W}_h$ are the solutions of the problems (Q) and ($Q_h$), we have $\forall \boldsymbol{v}_h \in \boldsymbol{V}_h$,

$$|b(\boldsymbol{v}_h, \Psi - \Psi_h)| = |A(\boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{v}_h)| \leqslant \|A\| \|\boldsymbol{u} - \boldsymbol{u}_h\|_V \|\boldsymbol{v}_h\|_V$$

(by continuity of $A(\cdot, \cdot)$)

$\Rightarrow \forall \Phi_h \in \mathbf{W}_h$,

$$\begin{aligned}\|\Psi_h - \Phi_h\|_{0,\Omega} &\leqslant \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{b(\boldsymbol{v}_h, \Psi - \Phi_h)}{\|\boldsymbol{v}_h\|_{V_p}} + \|A\| \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{\|\boldsymbol{u} - \boldsymbol{u}_h\|_V \|\boldsymbol{v}_h\|_V}{\|\boldsymbol{v}_h\|_{V_p}} \\ &\leqslant \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{b(\boldsymbol{v}_h, \Psi - \Phi_h)}{\|\boldsymbol{v}_h\|_{V_p}} + \|A\| \sigma_1 \|\boldsymbol{u} - \boldsymbol{u}_h\|_V \quad \text{(by (3.7))}.\end{aligned} \tag{6.10}$$

From (6.8), $\forall \Phi_h \in \mathbf{W}_h$,

$$\|\Psi - \Psi_h\|_{0,\Omega} \leqslant \|\Psi - \Phi_h\|_{0,\Omega} + \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{b(\boldsymbol{v}_h, \Psi - \Phi_h)}{\|\boldsymbol{v}_h\|_{V_p}} + \|A\| \sigma_1 \|\boldsymbol{u} - \boldsymbol{u}_h\|_V$$

$$\begin{aligned}\Rightarrow \|\Psi - \Psi_h\|_{0,\Omega} \leqslant{}& \inf_{\Phi_h \in \mathbf{W}_h} \|\Psi - \Phi_h\|_{0,\Omega} + \inf_{\Phi_h \in \mathbf{W}_h} \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{b(\boldsymbol{v}_h, \Psi - \Phi_h)}{\|\boldsymbol{v}_h\|_{V_p}} \\ &+ \|A\| \sigma_1 \|\boldsymbol{u} - \boldsymbol{u}_h\|.\end{aligned} \tag{6.11}$$

But

$$\inf_{\Phi_h \in \mathbf{W}_h} \sup_{v_h \in V_h} \frac{b(v_h, \Psi - \Phi_h)}{\| v_h \|_{V_p}} \leqslant \sup_{v_h \in V_h} \frac{b(v_h, \Psi - \Pi_h \Psi)}{\| v_h \|_{V_p}}$$

$$\leqslant \sigma_1 \sup_{v_h \in V_h} \frac{b(v_h, \Psi - \Pi_h \Psi)}{\| v_h \|_{V}}$$

$$\leqslant \sigma_1 \| \Psi - \Pi_h \Psi \|_{0,\Omega} \quad \text{(by (6.7))}$$

and

$$\inf_{\Phi_h \in \mathbf{W}_h} \| \Psi - \Phi_h \|_{0,\Omega} \leqslant \| \Psi - \Pi_h \Psi \|_{0,\Omega}$$

$$\Rightarrow \| \Psi - \Psi_h \|_{0,\Omega} \leqslant \sigma_1 \| A \| \, \| u - u_h \|_V + (1 + \sigma_1) \| \Psi - \Pi_h \Psi \|_{0,\Omega},$$

from which the result (6.2) follows with $\bar{c}_2 = \max\{ \sigma_1 \| A \|, 1 + \sigma_1 \}$. $\square$

Now we state the final result:

**Theorem 6.2.** *Let the solution $u \in H_0^2(\Omega)$ of* ($\mathrm{P_G}$) *belong to $H^{m+2}(\Omega)$, $m \geqslant 1$ and the coefficients $a_{ijkl} \in W^{m,\infty}(\Omega)$ $\forall i, j, k, l = 1, 2$. Then, for $m \geqslant 1$, $\exists$ a constant $C > 0$, independent of $h$, such that*

$$\| u - u_h \|_{1,\Omega} + \| \kappa^* - \kappa_h^* \|_{0,\Omega} + \| \Psi - \Psi_h \|_{0,\Omega} \leqslant C h^m ( \| u \|_{m+2,\Omega} + | \Psi |_{m,\Omega} ), \tag{6.12}$$

*where $((u, \kappa^*), \Psi) = (u, \Psi) \in V_p \times \mathbf{W}$, $(p > 2)$ and $((u_h, \kappa_h^*), \Psi_h) = (u_h, \Psi_h) \in V_h \times \mathbf{W}_h$ are the solutions of* (Q) *and* ($\mathrm{Q}_h$) *respectively.*

**Proof.** Let $u \in H_0^2(\Omega) \cap H^{m+2}(\Omega)$, be the solution of ($\mathrm{P}_G$) and $a_{ijkl} \in W^{m,\infty}(\Omega)$ $\forall i, j = 1, 2$, $m \geqslant 1$. Then, $\psi_{ij} = a_{ijkl} u_{,kl} \in H^m(\Omega)$ [11, p. 192]

$$\| \Psi \|_{m,\Omega} \leqslant C \| \kappa^* \|_{m,\Omega} \leqslant C \| u \|_{m+2,\Omega} \tag{6.13}$$

Then, from Theorem (3.5), $(u, \Psi) = ((u, \kappa^*), \Psi) \in V_p \times \mathbf{W}$ is the solution of (Q) with $u \in H^{m+2}(\Omega) \cap H_0^2(\Omega)$, $\kappa^* \in (H^m(\Omega))^4$, $\Psi \in (H^m(\Omega))^4$. From Lemma (4.4), we have

$$\| u - \Pi_{0h} u \|_{1,\Omega} \leqslant C_6 h^m | u |_{m+1,\Omega},$$

$$\| \kappa^* - \underset{\sim}{\Pi}_h \kappa^* \|_{0,\Omega} \leqslant C_4 h^m | \kappa^* |_{m,\Omega}, \qquad \| \Psi - \Pi_h \Psi \|_{0,\Omega} \leqslant C_2 h^m | \Psi |_{m,\Omega}. \tag{6.14}$$

Since

$$\| u - \Pi_h u \|_V^2 = \| u - \Pi_{0h} u \|_{1,\Omega}^2 + \| \kappa^* - \underset{\sim}{\Pi}_h \kappa^* \|_{0,\Omega}^2,$$

using (6.14), we get

$$\| u - \Pi_h u \|_V^2 \leqslant C_7 h^{2m} ( | u |_{m+1,\Omega}^2 + | \kappa^* |_{m,\Omega}^2 ), \quad C_7 > 0,$$

$$\Rightarrow \| u - \Pi_h u \|_V \leqslant C_8 h^m \| u \|_{m+2,\Omega}, \quad C_8 > 0. \tag{6.15}$$

From (6.1), (6.14) and (6.15), we get

$$\| u - u_h \|_V \leqslant C_9 h^m ( \| u \|_{m+2,\Omega} + | \Psi |_{m,\Omega} ), \quad C_9 > 0. \tag{6.16}$$

$$\| \Psi - \Psi_h \|_{0,\Omega} \leqslant C_{10} h^m [ ( \| u \|_{m+2,\Omega} + | \Psi |_{m,\Omega} ) + | \Psi |_{m,\Omega} ]$$

$$\leqslant C_{11}h^m(\|u\|_{m+2,\Omega} + |\Psi|_{m,\Omega}), \quad C_{10}, C_{11} > 0. \tag{6.17}$$

But

$$\|u - u_h\|_{1,\Omega} + \|\kappa^* - \kappa_h\|_{0,\Omega} \leqslant \sqrt{2}\left(\|u - u_h\|^2_{1,\Omega} + \|\kappa^* - \kappa^*_h\|^2_{0,\Omega}\right)^{1/2}$$
$$= \sqrt{2}\,\|\boldsymbol{u} - \boldsymbol{u}_h\|_V. \tag{6.18}$$

Now, the result (6.12) follows from (6.18), (6.16) and (6.17). □

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